Strong Forcing

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This is a streamlined exposition of the basic facts about forcing. It replaces Chapter VII, Section 3, pages 192–204, in Kunen's book. We follow the exposition in Shoenfield's paper "Unramified Forcing".

As usual, M is a countable transitive model of ZFC, P is a partial ordering in M, and M^P is the set of P-names. For any M-generic filter $G \subseteq P$ we have $M[G] = \{\tau_G : \tau \in M^P\}$ where $\tau_G = \{\sigma_G : \langle \sigma, p \rangle \in \tau, p \in G\}$.

Lemma 1. For any p there exists an M-generic filter $G \subseteq P$ such that $p \in G$.

Proof. This is easily proved, using countability of M. $\square$

Definition 2. The *forcing language* consists of the language of ZFC plus constant symbols τ for all $\tau \in M^P$. If φ is a sentence of the forcing language, $M[G] \models \varphi$ means that φ is true in $M[G]$ interpreting τ as τ_G.

Definition 3. Let $p \in P$, and let φ be a sentence of the forcing language. We define $p \Vdash \varphi$ (p forces φ) to mean that $M[G] \models \varphi$ for all M-generic filters $G \subseteq P$ such that $p \in G$.

Theorem 4 (definability of forcing). For any formula $\varphi(x_1, \ldots, x_n)$ we have that $\{\langle p, \tau_1, \ldots, \tau_n \rangle : p \Vdash \varphi(\tau_1, \ldots, \tau_n)\}$ is definable over M.

Theorem 5 (forcing equals truth). For all M-generic filters $G \subseteq P$, $M[G] \models \varphi$ if and only if $(\exists p \in G)\,(p \Vdash \varphi)$.

In order to prove Theorems 4 and 5, we introduce the notion of *strong forcing*. We assume that the forcing language has been set up with $\in$, $\neq$, $\neg$, $\vee$, $\exists$ as primitives. We define $x \notin y$ as $\neg(x \in y)$, and $x = y$ as $\neg(x \neq y)$.

Definition 6. We define $p \Vdash_s \varphi$ (p strongly forces φ) as follows.

1. $p \Vdash_s \sigma \in \tau$ if and only if, for some $q \geq p$, $\langle \sigma', q \rangle \in \tau$ for some σ' such that $p \Vdash_s \sigma = \sigma'$.

2. $p \Vdash_s \tau_1 \neq \tau_2$ if and only if, for some $q \geq p$ and some σ, either $\langle \sigma, q \rangle \in \tau_1$ and $p \Vdash_s \sigma \notin \tau_2$, or $\langle \sigma, q \rangle \in \tau_2$ and $p \Vdash_s \sigma \notin \tau_1$.

3. $p \Vdash_s \neg\varphi$ if and only if there does not exist $q \leq p$ such that $q \Vdash_s \varphi$.

4. $p \Vdash_s \varphi \vee \psi$ if and only if $p \Vdash_s \varphi$ or $p \Vdash_s \psi$.
5. $p \Vdash_s \exists x\, \varphi(x)$ if and only if $p \Vdash_s \varphi(\tau)$ for some τ.

Note that, for clauses 1 and 2, the definition is by transfinite induction on the ranks of σ, τ_1, and τ_2 as P-names. For clauses 3, 4, and 5, the definition is by induction on the rank of φ as a sentence of the forcing language.

Lemma 7. If $p \Vdash_s \varphi$ and $q \leq p$ then $q \Vdash_s \varphi$.

Lemma 8 (definability of strong forcing). For any formula $\varphi(x_1, \ldots, x_n)$ we have that $\{\langle p, \tau_1, \ldots, \tau_n\rangle : p \Vdash_s \varphi(\tau_1, \ldots, \tau_n)\}$ is definable over M.

Lemmas 7 and 8 are easily proved by induction, following the definition of $\Vdash_s$.

Lemma 9 (strong forcing equals truth). For all M-generic filters $G \subseteq P$, $M[G] \models \varphi$ if and only if $(\exists p \in G)\,(p \Vdash_s \varphi)$.

Proof. The proof is by induction, following the definition of $\Vdash_s$.

1. "if". Suppose $p \in G$ and $p \Vdash_s \sigma \in \tau$. By definition there exist $q \geq p$ and $\langle \sigma', q\rangle \in \tau$ such that $p \Vdash_s \sigma = \sigma'$. Then $q \in G$, hence $\sigma'_G \in \tau_G$. Also, by inductive hypothesis, $\sigma_G = \sigma'_G$. Hence $\sigma_G \in \tau_G$.

   "only if". Suppose $\sigma_G \in \tau_G$. By definition there exists $\langle \sigma', q\rangle \in \tau$ such that $\sigma_G = \sigma'_G$ and $q \in G$. By inductive hypothesis, there exists $r \in G$ such that $r \Vdash_s \sigma = \sigma'$. Since $q, r \in G$ there exists $p \in G$ such that $p \leq q, r$. By Lemma 7 we have that $p \Vdash_s \sigma = \sigma'$. Thus $p \Vdash_s \sigma \in \tau$.

2. "if". Suppose $p \in G$ and $p \Vdash \tau_1 \neq \tau_2$. Say $q \geq p$, $\langle \sigma, q\rangle \in \tau_1$, $p \Vdash_s \sigma \notin \tau_2$. Then $q \in G$, hence $\sigma_G \in \tau_{1G}$. Also, by inductive hypothesis, $\sigma_G \notin \tau_{2G}$. Thus $\tau_{1G} \neq \tau_{2G}$.

   "only if". Suppose $\tau_{1G} \neq \tau_{2G}$. Say $\langle \sigma, q\rangle \in \tau_1$, $q \in G$, $\sigma_G \notin \tau_{2G}$. By inductive hypothesis, there exists $r \in G$ such that $r \Vdash_s \sigma \notin \tau_2$. Since $q, r \in G$ there exists $p \in G$ such that $p \leq q, r$. By Lemma 7 we have that $p \Vdash_s \sigma \notin \tau_2$. Thus $p \Vdash_s \tau_1 \neq \tau_2$.

3. "if". Suppose $p \in G$ and $p \Vdash_s \neg\varphi$. To show $M[G] \models \neg\varphi$. Suppose $M[G] \models \varphi$. By inductive hypothesis, there exists $q \in G$ such that $q \Vdash_s \varphi$. Since $p, q \in G$, they are compatible, so let $r \leq p, q$. Then, by Lemma 7, $r \Vdash_s \varphi$, and $r \leq p$, contradicting $p \Vdash_s \neg\varphi$.

   "only if". Suppose $M[G] \models \neg\varphi$. Put $D = \{p : p \Vdash_s \varphi \text{ or } p \Vdash_s \neg\varphi\}$. Using the definition of $p \Vdash_s \neg\varphi$, it is easy to see that D is dense. Let $p \in D \cap G$. If $p \Vdash_s \varphi$, then by inductive hypothesis, $M[G] \models \varphi$, a contradiction. Hence $p \Vdash_s \neg\varphi$.

4. "if". Suppose $p \in G$ and $p \Vdash_s \varphi \vee \psi$. Say $p \Vdash_s \varphi$. By inductive hypothesis, $M[G] \models \varphi$. Hence $M[G] \models \varphi \vee \psi$.

   "only if". Suppose $M[G] \models \varphi \vee \psi$. Say $M[G] \models \varphi$. By inductive hypothesis, there exists $p \in G$ such that $p \Vdash_s \varphi$. Hence $p \Vdash_s \varphi \vee \psi$.

5. "if". Suppose $p \in G$ and $p \Vdash_s \exists x\, \varphi(x)$. Then $p \Vdash_s \varphi(\tau)$ for some τ. By inductive hypothesis, $M[G] \models \varphi(\tau)$. Hence $M[G] \models \exists x\, \varphi(x)$.

   "only if". Suppose $M[G] \models \exists x\, \varphi(x)$. Then $M[G] \models \varphi(\tau)$ for some τ. By inductive hypothesis, there exists $p \in G$ such that $p \Vdash_s \varphi(\tau)$. Then $p \Vdash_s \exists x\, \varphi(x)$.

This completes the proof.

□

Lemma 10. $p \Vdash \varphi$ if and only if $\{r \leq p : r \Vdash_s \varphi\}$ is dense below p.

Proof. "if". Assume that $\{r \leq p : r \Vdash_s \varphi\}$ is dense below p. To show $p \Vdash \varphi$. Let G be generic with $p \in G$. Then there exists $r \in G$ such that $r \Vdash_s \varphi$. Hence, by Lemma 9, $M[G] \models \varphi$.

"only if". Assume $p \Vdash \varphi$. To show that $\{r \leq p : r \Vdash_s \varphi\}$ is dense below p. Given $q \leq p$, by Lemma 1 let G be generic with $q \in G$. Then $p \in G$, hence $M[G] \models \varphi$. By Lemma 9 there exists $p' \in G$ such that $p' \Vdash_s \varphi$. Since $p', q \in G$, they are compatible, so let $r \leq p', q$. Then, by Lemma 7, $r \Vdash_s \varphi$.

This completes the proof. □

Theorems 4 and 5 follow easily from Lemmas 8, 9, and 10.

Corollary 11.

1. If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.
2. $p \Vdash \neg\varphi$ if and only if there does not exist $q \leq p$ such that $q \Vdash \varphi$.
3. If $p \Vdash \varphi \vee \psi$ then there exists $q \leq p$ such that $q \Vdash \varphi$ or $q \Vdash \psi$.
4. If $p \Vdash \exists x\, \varphi(x)$ then there exists $q \leq p$ such that $q \Vdash \varphi(\tau)$ for some τ.