

# Math 558 – Homework #2

Due October 5, 2009

## Solutions

1. Prove that the following sets are not recursive.

- (a)  $T = \{e \mid \varphi_e^{(1)} \text{ is total}\}.$
- (b)  $E = \{e \mid \varphi_e^{(1)} \text{ is empty}\}.$
- (c)  $F = \{e \mid \text{dom}(\varphi_e^{(1)}) \text{ is finite}\}.$
- (d)  $O = \{e \mid 1 \in \text{rng}(\varphi_e^{(1)})\}.$
- (e)  $I = \{3^i 5^j \mid \varphi_i^{(1)} = \varphi_j^{(1)}\}.$

*Solution.* By the Parametrization Theorem, let  $f(m)$  be a primitive recursive function such that  $\varphi_{f(m)}^{(1)}(n) \simeq 1 + 0 \cdot \varphi_m^{(1)}(0)$  for all  $m, n$ . Recall that  $H$  is the Halting Problem,  $H = \{m \mid \varphi_m^{(1)}(0) \downarrow\}$ . For all  $m \in H$  we have  $f(m) \in T$ ,  $f(m) \notin E$ ,  $f(m) \notin F$ , and  $f(m) \in O$ . Also, for all  $m \notin H$  we have  $f(m) \notin T$ ,  $f(m) \in E$ ,  $f(m) \in F$ , and  $f(m) \notin O$ . Thus  $f$  reduces  $H$  to  $T$ ,  $E^c$ ,  $F^c$ , and  $O$ . Since  $H$  is nonrecursive, it follows that  $T, E, F, O$  are nonrecursive.

Fix  $i$  such that  $i \in E$ , i.e.,  $\varphi_i^{(1)}(n) \uparrow$  for all  $n$ . Then  $E$  is reducible to  $I$  via the primitive recursive function  $j \mapsto 3^i 5^j$ . Since  $E$  is nonrecursive, it follows that  $I$  is nonrecursive.

2. Prove the following theorem, which generalizes both the Parametrization Theorem and the Recursion Theorem.

**Theorem.** Let  $\theta(e, m_1, \dots, m_k, n_1, \dots, n_l)$  be a  $k + l + 1$ -place partial recursive function. Then, we can find a  $k$ -place primitive recursive function  $f(m_1, \dots, m_k)$  such that

$$\varphi_{f(m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_l) \simeq \theta(f(m_1, \dots, m_k), m_1, \dots, m_k, n_1, \dots, n_l)$$

for all  $m_1, \dots, m_k, n_1, \dots, n_l$ .

*Solution.* First we generalize the Parametrization Theorem, as follows.

**Lemma.** Given a  $k + l$ -place partial recursive function  $\psi$ , we can find a  $k$ -place primitive recursive function  $f$  such that

$$\varphi_{f(m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_l) \simeq \psi(m_1, \dots, m_k, n_1, \dots, n_l)$$

for all  $m_1, \dots, m_k, n_1, \dots, n_l$ .

*Proof.* Let  $\psi$  be a  $k + l$ -place partial recursive function. By the Parametrization Theorem, let  $f^*$  be a 1-place primitive recursive function such that

$$\varphi_{f^*(m)}^{(l)}(n_1, \dots, n_l) \simeq \psi((m)_1, \dots, (m)_k, n_1, \dots, n_l)$$

for all  $m, n_1, \dots, n_l$ . The idea here is that  $m$  encodes the finite sequence  $(m)_1, \dots, (m)_k$  via prime power coding. Letting

$$f(m_1, \dots, m_k) = f^*(p_1^{m_1} \cdots p_k^{m_k})$$

we see that  $f$  is a  $k$ -place primitive recursive function and

$$\begin{aligned} \varphi_{f(m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_l) &\simeq \varphi_{f^*(p_1^{m_1} \cdots p_k^{m_k})}^{(l)}(n_1, \dots, n_l) \\ &\simeq \psi^*(p_1^{m_1} \cdots p_k^{m_k}, n_1, \dots, n_l) \\ &\simeq \psi(m_1, \dots, m_k, n_1, \dots, n_l) \end{aligned}$$

for all  $m_1, \dots, m_k, n_1, \dots, n_l$ . This proves the lemma.

Now, to prove our theorem, we imitate the proof of the Recursion Theorem. Let  $\theta$  be a  $k + l + 1$ -place partial recursive function. By our lemma, let  $g$  be a  $k + 1$ -place primitive recursive function such that

$$\varphi_{g(e, m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_l) \simeq \theta(e, m_1, \dots, m_k, n_1, \dots, n_l)$$

for all  $e, m_1, \dots, m_k, n_1, \dots, n_l$ . By the Parametrization Theorem, let  $d$  be a 1-place primitive recursive function such that

$$\varphi_{d(u)}^{(l)}(n_1, \dots, n_l) \simeq \varphi_{\varphi_u^{(1)}(u)}^{(l)}(n_1, \dots, n_l)$$

for all  $u, n_1, \dots, n_l$ . By our lemma, let  $h$  be a  $k$ -place primitive recursive function such that

$$\varphi_{h(m_1, \dots, m_k)}^{(1)}(u) \simeq g(d(u), m_1, \dots, m_k)$$

for all  $m_1, \dots, m_k, u$ . Finally, letting

$$f(m_1, \dots, m_k) = d(h(m_1, \dots, m_k))$$

we see that  $f$  is a  $k$ -place primitive recursive function and

$$\begin{aligned} \varphi_{f(m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_l) &\simeq \varphi_{d(h(m_1, \dots, m_k))}^{(l)}(n_1, \dots, n_l) \\ &\simeq \varphi_{\varphi_{h(m_1, \dots, m_k)}^{(1)}(h(m_1, \dots, m_k))}^{(l)}(n_1, \dots, n_l) \\ &\simeq \varphi_{g(d(h(m_1, \dots, m_k)), m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_k) \\ &\simeq \varphi_{g(f(m_1, \dots, m_k), m_1, \dots, m_k)}^{(l)}(n_1, \dots, n_k) \\ &\simeq \theta(f(m_1, \dots, m_k), m_1, \dots, m_k, n_1, \dots, n_k) \end{aligned}$$

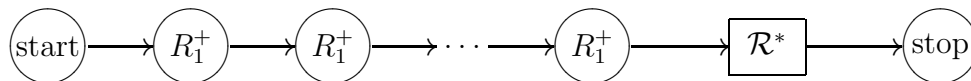
for all  $m_1, \dots, m_k, n_1, \dots, n_l$ . This proves the theorem.

3. We know that the Halting Problem is unsolvable. In other words, the set of Gödel numbers of register machine programs which eventually halt is nonrecursive. Prove the same result for register machine programs using only two registers,  $R_1$  and  $R_2$ .

*Solution.* Let  $H_2$  be the Halting Problem for register machine programs which use only  $R_1$  and  $R_2$ . In other words,  $H_2 = \{\#(\mathcal{P}) \mid \mathcal{P} \text{ uses only } R_1 \text{ and } R_2 \text{ and } \mathcal{P}(0) \text{ eventually halts}\}$ . We shall show that  $H$ , the Halting Problem for arbitrary register machine programs, is reducible to  $H_2$ . Since  $H$  is nonrecursive, it will follow that  $H_2$  is nonrecursive.

As a special case of Exercise 1.5.10 in the lecture notes, let  $\mathcal{R}$  be a register machine program using only  $R_1$  and  $R_2$  which computes the 1-place partial recursive function  $3^n \mapsto 5^{\varphi_n^{(1)}(0)}$ . In particular  $\mathcal{R}(3^n)$  eventually halts if and only if  $\varphi_n^{(1)}(0) \downarrow$ , i.e.,  $n \in H$ . If  $I_1, \dots, I_l$  are the instructions of  $\mathcal{R}$ , let  $\mathcal{R}^*$  be  $\mathcal{R}$  with the instructions renumbered as  $I_2, \dots, I_{l+1}$ . For each  $n$  let  $\mathcal{R}_n^*$  be the program

$$I_1 \quad I_{l+2} \quad \cdots \quad I_{l+3^n} \quad I_2 \cdots I_{l+1}$$



which is like  $\mathcal{R}$  but with  $3^n$  hard-coded as the input. Thus  $\mathcal{R}_n^*(0)$  eventually halts if and only if  $\mathcal{R}(3^n)$  eventually halts, i.e.,  $n \in H$ . As in the proof of the Parametrization Theorem (see the lecture notes, Theorem 1.7.10), we can show that  $f(n) = \#(\mathcal{R}_n^*)$  is primitive recursive as a function of  $n$ . Thus  $f$  reduces  $H$  to  $H_2$ , Q.E.D.