Math 558 – Homework #1

Due September 15, 2009

Solutions

1. A real number α is said to be *primitive recursive* if the function $f(n) =$ the nth digit of α is primitive recursive. A real number α is said to be *algebraic* if it is a root of a nonzero polynomial with integer coefficients. For example, $\sqrt{2}$ is a real algebraic number, because it is a root of the polynomial $x^2 - 2$.

   Prove that all real algebraic numbers are primitive recursive.

   Solution. Let $p(x)$ be a nonzero polynomial of minimal degree with integer coefficients such that $p(\alpha) = 0$. Then α is a simple root of $p(x)$, i.e., $p'(\alpha) \neq 0$ where $p'(x)$ is the derivative of $p(x)$. Without loss of generality, assume that $\alpha > 0$ and $p'(\alpha) > 0$ and α is irrational. Let a and b be rational numbers such that $0 < a < \alpha < b$ and $p(x)$ is negative for $a < x < \alpha$ and positive for $\alpha < x < b$. Let m be such that, for all sufficiently large n, the $(m + n)$th digit of α is Remainder$(g(n), 10)$ where $g(n)$ is the least k such that

   $$a < \frac{k}{10^n} < b \text{ and } p\left(\frac{k+1}{10^n}\right) > 0.$$

   Here we are applying the bounded least number operator, so g is primitive recursive. From this it follows easily that f is primitive recursive, where $f(n) =$ the nth digit of α.

2. We know that the Ackermann function is an example of a 1-place function which is recursive but not primitive recursive. Find an example of a 1-place predicate which is recursive but not primitive recursive.

   Solution. Our example will be obtained by diagonalizing over all 1-place primitive recursive functions. We first prove the following lemma and theorem.

Lemma. We can find a 2-place total recursive function $r(e,n)$ with the following property. For all $k \geq 1$ and all k-place primitive recursive functions f, there exists e such that $r(e, \prod_{i=1}^{k} p_i^{n_i}) = f(n_1, \ldots, n_k)$ for all $n_1, \ldots, n_k$.

Proof. Consider a system of *indices* for the primitive recursive functions, defined inductively as follows.

(a) Let 2 be an index for the constant zero function, $Z(m) = 0$.

(b) Let 2^2 be an index for the successor function, $S(m) = m + 1$.

(c) For $1 \leq i \leq k$ let $2^3 \cdot 3^k \cdot 5^i$ be an index of the k-place projection function $P_{ki}(n_1, \ldots, n_k) = n_i$.

(d) If $u_1, \ldots, u_l, v$ are indices of $g_1(n_1, \ldots, n_k)$, $\ldots$, $g_l(n_1, \ldots, n_k)$, $h(t_1, \ldots, t_l)$ respectively, let $2^4 \cdot 3^k \cdot 5^v \cdot \prod_{j=1}^{l} p_{j+2}^{u_j}$ be an index of $f(n_1, \ldots, n_k)$ given by generalized composition as

$$f(n_1, \ldots, n_k) = h(g_1(n_1, \ldots, n_k), \ldots, g_l(n_1, \ldots, n_k)).$$

(e) If u and v are indices of $g(n_1, \ldots, n_k)$ and $h(m, t, n_1, \ldots, n_k)$ respectively, let $2^5 \cdot 3^{k+1} \cdot 5^v \cdot 7^u$ be an index of $f(m, n_1, \ldots, n_k)$ given by primitive recursion as

$$\begin{aligned} f(0, n_1, \ldots, n_k) &= g(n_1, \ldots, n_k), \\ f(m+1, n_1, \ldots, n_k) &= h(m, f(m, n_1, \ldots, n_k), n_1, \ldots, n_k). \end{aligned}$$

It is routine to show that the set of indices is primitive recursive. Moreover, if e is an index of a k-place primitive recursive function, then $(e)_1 = k$. By the Recursion Theorem, let $\psi(e,m)$ be a 2-place partial recursive function with the following properties. Writing $n = \prod_{i=1}^{k} p_i^{n_i}$ and $n' = \prod_{i=1}^{k} p_{i+1}^{n_i}$ and $n'' = \prod_{i=1}^{k} p_{i+2}^{n_i}$ we have:

(a) $\psi(2, p_1^m) \simeq 0$.

(b) $\psi(2^2, p_1^m) \simeq m + 1$.

(c) $\psi(e, n) \simeq n_i$ whenever e is an index of the form $2^3 \cdot 3^k \cdot 5^i$.

(d) $\psi(e, n) \simeq \psi(v, \prod_{j=1}^{l} p_j^{\psi(u_j, n)})$ whenever e is an index of the form $2^4 \cdot 3^k \cdot 5^v \cdot \prod_{j=1}^{l} p_{j+2}^{u_j}$.

(e) $\psi(e, p_1^0 \cdot n') \simeq \psi(u, n)$ and $\psi(e, p_1^{m+1} \cdot n') \simeq \psi(v, p_1^m \cdot p_2^{\psi(e, p_1^m \cdot n')} \cdot n'')$ whenever e is an index of the form $2^5 \cdot 3^{k+1} \cdot 5^v \cdot 7^u$.

(f) $\psi(e, m) \simeq 0$ otherwise.

By induction on e with a subsidiary induction on m, it is straightforward to prove that $\psi(e, m) \downarrow$ for all e and m, and that $\psi(e, \prod_{i=1}^k p_i^{n_i}) = f(n_1, \ldots, n_k)$ whenever e is an index of a k-ary primitive recursive function f. Letting $r(e, m) = \psi(e, m)$ we have our lemma.

Theorem. For each $k \geq 1$ we can find a $k+1$-place total recursive function r_k with the following property. For any k-place primitive recursive function f there exists e such that $r_k(e, n_1, \ldots, n_k) = f(n_1, \ldots, n_k)$ for all $n_1, \ldots, n_k$.

Proof. Let $r_k(e, n_1, \ldots, n_k) = r(e, \prod_{i=1}^k p_i^{n_i})$ where r is as in the lemma. Clearly r_k has the desired property.

Now, define $d : \mathbb{N} \to \{0, 1\}$ by $d(n) = 1 \dot{-} r_1(n, n)$. Clearly d is the characteristic function of a 1-place predicate which is recursive but not primitive recursive.

3. Exhibit a register machine program showing that the function $f(m, n) = m^n$ is computable. (Note that $m^0 = 1$ for all $m \in \mathbb{N}$ including $m = 0$. This convention makes the recursion easier.)

Solution.