Math 558 – Homework #4

Due December 10, 2009

1. Define ordinal exponentiation by transfinite recursion as follows.

$$\begin{array}{lcl} \alpha^0 & = & 1, \\ \alpha^{\beta+1} & = & \alpha^\beta \cdot \alpha, \\ \alpha^\delta & = & \sup_{\beta<\delta} \alpha^\beta \end{array}$$

   for all limit ordinals δ. In this exercise we shall obtain a more explicit representation of α^β.

   Let $\alpha = \text{type}((A, R))$ and $\beta = \text{type}((B, S))$. Let a_0 be the R-least element of A. Let C be the set of functions $f : B \to A$ such that $f(b) = a_0$ for all but finitely many $b \in B$. Let T be the ordering of C by last difference (see Exercise 4.4.4 in the lecture notes). Show that $\alpha^\beta = \text{type}((C, T))$.

2. Let X be a set of sets. By a *chain within* X we mean a set $C \subseteq X$ such that for all $U, V \in C$ either $U \subseteq V$ or $V \subseteq U$. A chain within X is said to be *maximal* if it is not properly included in any other chain within X.

   Use the Axiom of Choice plus transfinite recursion to prove that there exists a maximal chain within X.

   Note: This is a version of Zorn's Lemma.

3. Let λ be an infinite cardinal. The *cofinality of* λ, abbreviated $\text{cf}(\lambda)$, is defined to be the least κ such that λ can be written as the sum of κ-many cardinals each less than λ. Note that $\text{cf}(\lambda) \leq \lambda$.

   (a) Show that $\text{cf}(\lambda)$ is an infinite regular cardinal.

   (b) Show that for any infinite regular cardinal κ there exist arbitrarily large strong limit cardinals λ such that $\text{cf}(\lambda) = \kappa$.

   (c) Show that λ is regular if and only if $\text{cf}(\lambda) = \lambda$.

   (d) For any infinite cardinal κ, show that $\text{cf}(\mu^\kappa) > \kappa$ for all $\mu \geq 2$. Two special cases of this are $\text{cf}(2^\kappa) > \kappa$ and $\lambda^{\text{cf}(\lambda)} > \lambda$.

   (e) Assuming the GCH, prove that for all infinite cardinals κ and λ we have

$$\lambda^\kappa = \begin{cases} \lambda & \text{if } \kappa < \text{cf}(\lambda), \\ \lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa \leq \lambda, \\ \kappa^+ & \text{if } \kappa \geq \lambda. \end{cases}$$