Computability, Unsolvability, Randomness
Math 497A: Homework #12

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Due Monday, November 26, 2007

1. (a) Let P be a Π^0_1 subset of $2^{\mathbb{N}}$. If P has only finitely many elements, prove that all of the elements of P are recursive.

   Hint: Use Lemma 48.3, a.k.a., the Magic Lemma.

   (b) Does this hold with $\mathbb{N}^{\mathbb{N}}$ instead of $2^{\mathbb{N}}$?

2. (a) Let $P \subseteq 2^{\mathbb{N}}$ be nonempty Π^0_1 with no recursive elements. Prove that for all Y we can find $X \in P$ such that $X' \equiv_T X \oplus 0' \equiv_T Y \oplus 0'$.

   Note: This result is a combination of the Low Basis Theorem and the Friedberg Jump Inversion Theorem. The proof is basically a combination of the two proofs.

   (b) Deduce that for all Y we can find a random X such that $X' \equiv_T X \oplus 0' \equiv_T Y \oplus 0'$.

3. Recall that we have defined

$$K(n) = K(\underbrace{\langle 1, \ldots, 1 \rangle}_{n})$$

   for all $n \in \mathbb{N}$.

   Prove that

$$K(\tau) \le C(\tau) + K(C(\tau)) + O(1)$$

   for all bitstrings τ.

4. Assume that $f(n)$ is a recursive function such that

$$\sum_{n=0}^{\infty} \frac{1}{2^{f(n)}} \ < \ \infty \ .$$

   Prove that $K(n) \le f(n) + O(1)$ for all n.