Computability, Unsolvability, Randomness
Math 497A: Homework #11

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For $f, g \in \mathbb{N}^\mathbb{N}$ say that f is *majorized* by g if $f(n) < g(n)$ for all n.

1. If $P(f, g, -)$ is a Π^0_1 predicate, prove that the predicate

   $$Q(g, -) \;\equiv\; \exists f\, (P(f, g, -) \wedge f \text{ is majorized by } g)$$

   is again Π^0_1.

   Note: This is a generalization of the Magic Lemma, Lemma 48.3 in the Lecture Notes. You can prove it by imitating the the proof of the Magic Lemma.

2. (a) Show that the result of Problem 1 holds if we replace Π^0_1 by Σ^0_2.

   (b) Show that the result does not hold if we replace Π^0_1 by Π^0_2.
   In fact, we can find a Π^0_2 predicate $P(X, -)$ with X ranging over $2^\mathbb{N}$ such that the predicate $\exists X\, P(X, -)$ is not arithmetical, i.e., it is not Π^0_n or Σ^0_n for any n.

3. Let $P \subseteq 2^\mathbb{N}$ be Π^0_1. Let $\Phi(X, n)$ be a partial recursive functional such that $\Phi(X, n)\downarrow$ for all $X \in P$ and all n. Find a total recursive function $g(n)$ which majorizes $\Phi(X, n)$ for all $X \in P$ and all n.

4. An oracle X is said to be *hyperimmune-free* (sorry for the awkward terminology) if each $f \leq_T X$ is majorized by some recursive function.

   Note: This is another example of a "lowness property" of X.

   (a) Let $P \subseteq 2^\mathbb{N}$ be nonempty and Π^0_1. Prove that there exists $X \in P$ such that X is hyperimmune-free. This result is known as the Hyperimmune-Free Basis Theorem.
   Hint: Use Π^0_1 approximation as in the Low Basis Theorem.

   (b) Deduce that we can find a random X which is hyperimmune-free.

5. Prove that if $0 <_T X \leq_T 0'$ then X is not hyperimmune-free.

   Note: This prevents us from combining the Low Basis Theorem and the Hyperimmune-Free Basis Theorem into one theorem.

   Hint for the proof: By Post's Theorem X is Δ^0_2. Deduce that the singleton set $\{X\}$ is Π^0_2. Use this to find $f \equiv_T X$ such that the singleton set $\{f\}$ is Π^0_1. If such an f is majorized by a recursive function, use the result of Problem 1 to show that f is recursive.

6. (Extra Credit)

   (a) Prove that if X is 2-random then X is not hyperimmune-free.

   (b) What if we assume only that X is weakly 2-random?

7. (a) Prove that if Y is nonrecursive then $\mu(\{X \in 2^{\mathbb{N}} \mid Y \not\leq_T X\}) = 1$.

   (b) Deduce that for each nonrecursive Y we can find a random X such that $Y \not\leq_T X$.

   (c) More generally, prove the following. Given a sequence of nonrecursive oracles Y_i, $i = 0, 1, 2, \ldots$, we can find an X which is n-random for all n and such that $Y_i \not\leq_T X$ for all i.

   Note: It can be shown that for all Y we can find a random X such that $Y \leq_T X$. In fact, each Turing degree $\geq \mathbf{0}'$ contains a random X. However, this does not hold for weakly 2-random X's, because all such X's are Turing incomparable with $0'$.

8. (a) Assume that $P \subseteq 2^{\mathbb{N}}$ is Π^0_1 and

   $$\neg \exists X \, (X \in P \wedge X \text{ is recursive}).$$

   Find a nonrecursive Y such that

   $$\neg \exists X \, (X \in P \wedge X \leq_T Y).$$

   Hint: Use finite approximation.

   (b) Find a nonrecursive Y such that

   $$\neg \exists X \, (X \text{ is random} \wedge X \leq_T Y).$$

   Hint: Use the fact that $\{X \mid X \text{ is random}\}$ is the union of a sequence of Π^0_1 sets.