Computability, Unsolvability, Randomness
Math 497A: Homework #10

Stephen G. Simpson

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1. Let f and g be Turing oracles. Define $f \leq_{LK} g$ to mean that

$$K^g(\tau) \ \leq \ K^f(\tau) + O(1)$$

for all bitstrings τ. Define $f \leq_{LR} g$ to mean that

$$(\forall X \in 2^{\mathbb{N}})\,(\text{if } X \text{ is } g\text{-random then } X \text{ is } f\text{-random}).$$

   (a) Show that $f \leq_T g$ implies both $f \leq_{LK} g$ and $f \leq_{LR} g$.

   (b) Let $X \in 2^{\mathbb{N}}$ be such that $X \leq_{LK} 0$. Show that X is K-*trivial*, i.e., $K(X \upharpoonright n) \leq K(n) + O(1)$ for all n.

   Note: It can be shown that the properties $f \leq_{LK} g$ and $f \leq_{LR} g$ are equivalent to each other. However, they are not equivalent to $f \leq_T g$. In fact, we can find a nonrecursive $X \in 2^{\mathbb{N}}$ such that $X \leq_{LK} 0$. It can be shown that $X \leq_{LK} 0$ if and only if X is K-trivial.

2. For convenience in stating this problem, let us identify subsets of $\mathbb{N}$ with their characteristic functions. In other words, we identify $A \subseteq \mathbb{N}$ with $\chi_A \in 2^{\mathbb{N}}$. Thus $2^{\mathbb{N}}$ is the set of all subsets of $\mathbb{N}$.

   Let $J : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Turing jump operator:

$$J(X) = X' = H^X = \text{the Halting Problem relative to } X.$$

   Recall that $0^{(1)} = 0' = J(0)$ and in general $0^{(n+1)} = (0^{(n)})' = J(0^{(n)})$ for all n. By Post's Theorem we know that for each $n \geq 1$ the set $0^{(n)}$ is Σ^0_n and not Δ^0_n. Define

$$0^{(\omega)} \ = \ \bigoplus_{n=1}^{\infty} 0^{(n)} \ = \ \{3^m 5^n \mid m \in 0^{(n)}\}\ .$$

   Note that the set $0^{(\omega)}$ is not arithmetical, i.e., it is not Δ^0_n for any n.

(a) Show that the 2-place predicate $P \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ given by

$$P(X, Y) \quad \equiv \quad J(X) = Y$$

is Π^0_2.

(b) Show that for each $n \geq 1$ the singleton set $\{0^{(n)}\}$ is Π^0_2.

(c) Show that the singleton set $\{0^{(\omega)}\}$ is Π^0_2.

Note: These singleton sets are subsets of $2^{\mathbb{N}}$.

3. (a) Show that every nonempty Π^0_1 subset of $2^{\mathbb{N}}$ contains a member which is Δ^0_n for some n.

   (b) In part (a), what is the optimal value of n?

   (c) In parts (a) and (b), what if we replace Π^0_1 sets by Π^0_2 sets?

   (d) Is every Π^0_2 subset of $2^{\mathbb{N}}$ Turing isomorphic to a Π^0_1 subset of $2^{\mathbb{N}}$?

4. Let $X \in 2^{\mathbb{N}}$. We say that X is 2-*random* if X is random relative to $0'$. Recall also that X is *weakly* 2-*random* if $X \notin$ any Π^0_2 set of measure 0. Let $\mathbf{a} = \deg_T(X) =$ the Turing degree of X.

   (a) Show that if X is 2-random then X is weakly 2-random.

   (b) Show that if X is weakly 2-random then $\inf(\mathbf{a}, \mathbf{0}') = \mathbf{0}$.

   (c) In part (b) what if we assume only that X is random?

   (d) Show that if X is 2-random then $\sup(\mathbf{a}, \mathbf{0}') = \mathbf{a}'$.

   (e) In part (d) what if we assume only that X is weakly 2-random?

5. Show that every Π^0_2 subset of $2^{\mathbb{N}}$ includes a $\Sigma^{0,0'}_2$ set of the same measure.