Computability, Unsolvability, Randomness
Math 497A: Homework #9

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Due Monday, October 29, 2007

1. Hoeffding's Inequality says that the probability space $2^{\mathbb{N}}$ with the fair coin probability measure satisfies

$$\mathrm{Prob}\left(\left|\frac{\sum_{i=0}^{n-1} X(i)}{n} - \frac{1}{2}\right| > \epsilon\right) \quad < \quad \frac{2}{\exp 2n\epsilon^2}\ .$$

   Use Hoeffding's Inequality to prove that if a point $X \in 2^{\mathbb{N}}$ is *random* (i.e., random in the sense of Martin-Löf), then X obeys the Strong Law of Large Numbers:

$$\frac{\sum_{i=0}^{n-1} X(i)}{n} \to \frac{1}{2} \qquad \text{as} \qquad n \to \infty\,.$$

2. Prove that there exist weakly 1-random points in $2^{\mathbb{N}}$ which do not obey the Strong Law of Large Numbers.

   Hint: Use finite approximation.

3. In problem 1, can you say anything about the rate of convergence to 1/2?

4. Prove that if $X \oplus Y \in 2^{\mathbb{N}}$ is *random* (i.e., random in the sense of Martin-Löf), then $X \not\leq_T Y$ and $Y \not\leq_T X$.

5. Prove that there exist points $X, Y \in 2^{\mathbb{N}}$ such that $X \oplus Y$ is weakly 1-random yet $X \equiv_T Y$.

6. A set $B \subseteq \mathbb{N}$ is said to be *biimmune* if both B and its complement $\mathbb{N} \setminus B$ are immune. Prove that if $X \in 2^{\mathbb{N}}$ is weakly 1-random then X is the characteristic function of a biimmune set.

7. Let f be a Turing oracle.

For each $i \in \mathbb{N}$ define

$$U_i^f = \{X \in 2^{\mathbb{N}} \mid \varphi_i^{(1),f \oplus X}(0) \downarrow\}.$$

Thus U_i^f, $i = 0, 1, 2, \ldots$ is the standard recursive enumeration of all $\Sigma_1^{0,f}$ subsets of $2^{\mathbb{N}}$.

Given a sequence of sets $V_n \subseteq 2^{\mathbb{N}}$, $n = 0, 1, 2, \ldots$, prove that the following are pairwise equivalent.

(a) There exists a total recursive function g such that $V_n = U_{g(n)}^f$ for all n.

(b) There exists a total f-recursive function h such that $V_n = U_{h(n)}^f$ for all n.

(c) The predicate $P \subseteq 2^{\mathbb{N}} \times \mathbb{N}$ given by

$$P(X, n) \quad \equiv \quad X \in V_n$$

is $\Sigma_1^{0,f}$.

In this case we say that the sequence of sets V_n, $n = 0, 1, 2, \ldots$ is *uniformly* $\Sigma_1^{0,f}$ or *uniformly* Σ_1^0 *relative to* f.

Note: This concept will be part of the definition of what it means for a point $X \in 2^{\mathbb{N}}$ to be random relative to the oracle f.