Computability, Unsolvability, Randomness
Math 497A: Homework #8

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1. Let $K(\tau)$ denote the *prefix-free complexity* of a bitstring τ. Prove that

$$K(\tau_1{}^\frown\tau_2) \quad \leq \quad K(\tau_1) + K(\tau_2) + O(1) \ .$$

2. (a) Give an example of a subset of $\mathbb{N}^\mathbb{N}$ which is Σ^0_2 but not $\Sigma^{0,0'}_1$.

   (b) Can you replace $\mathbb{N}^\mathbb{N}$ by $2^\mathbb{N}$ here?

   Note: Recall Post's Theorem, which says (among other things) that a subset of $\mathbb{N}$ is Σ^0_2 if and only if it is $\Sigma^{0,0'}_1$. The point of (a) is to show that Post's Theorem does not hold for subsets of $\mathbb{N}^\mathbb{N}$.

   Hint: Recall that a set is open if and only if it is Σ^0_1 relative to an oracle. Therefore, it suffices to find a set which is Σ^0_2 and not open.

3. A real number is said to be *left recursively enumerable* (respectively *right recursively enumerable*) if it is the limit of an increasing (respectively decreasing) recursive sequence of rational numbers.

   (a) If A is a recursively enumerable subset of $\mathbb{N}$, show that the real number $\sum_{n\in A} 1/2^n$ is left recursively enumerable.

   (b) Show that there exist real numbers which are left recursively enumerable but not recursive.

   (c) Show that a real number is recursive if and only if it is both left recursively enumerable and right recursively enumerable.

4. Let P be a Π^0_1 subset of $2^\mathbb{N}$. We have seen how to construct a recursive tree $T \subseteq 2^{<\mathbb{N}}$ such that $P = \{\text{paths through } T\}$. For each $n = 0, 1, 2, \ldots$ let T_n be the set of strings in T of length n.

   (a) Show that T_n is prefix-free.

(b) Show that the set

$$V_n \;=\; \bigcup_{\tau \in T_n} N_\tau$$

is Δ^0_1. (Note that V_n is a subset of $2^{\mathbb{N}}$.)

(c) Show that P is the intersection of the V_n's. In other words,

$$P \;=\; \bigcap_{n=0}^{\infty} V_n \;.$$

(d) Show that the measure of P is given by

$$\mu(P) \;=\; \lim_{n \to \infty} \frac{|T_n|}{2^n} \;.$$

Here $|T_n|$ denotes the number of strings in T_n.

(e) Show that the real number $\mu(P)$ is right recursively enumerable.

(f) Show that $\mu(P)$ is not necessarily a recursive real number.

5. Given a nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, can we always find a member of P which is recursive?

   Hint: Consider a recursively inseparable pair of r.e. sets.

6. Two sets $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ are said to be *Turing isomorphic* if the members of P and Q have the same Turing degrees, i.e.,

$$\{\deg_T(f) \mid f \in P\} \;=\; \{\deg_T(g) \mid g \in Q\} \;.$$

(a) Prove that every Π^0_2 subset of $\mathbb{N}^{\mathbb{N}}$ is Turing isomorphic to a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$.

(b) Prove that every Π^0_2 subset of $\mathbb{N}^{\mathbb{N}}$ is Turing isomorphic to a Π^0_2 subset of $2^{\mathbb{N}}$.

(c) Is every Π^0_2 subset of $2^{\mathbb{N}}$ Turing isomorphic to a Π^0_1 subset of $2^{\mathbb{N}}$? Justify your answer.

Hints: (a) If $\forall x\, \exists y\, R(f, x, y)$ holds, map f to $f \oplus g$ where $g(x) = \mu y\, R(f, x, y)$. (b) Map f to the characteristic function of the set $G_f = \{3^x 5^y \mid f(x) = y\}$ = the "graph" of f.