Computability, Unsolvability, Randomness
Math 497A: Homework #7

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Due Monday, October 15, 2007

1. Exhibit an oracle program $\mathcal{P}$ such that
$$\varphi_e^{(1),f}(x) \quad \simeq \quad \mu y\,(y > x \wedge f(y) = 0)$$
for all $f \in \mathbb{N}^\mathbb{N}$ and all $x \in \mathbb{N}$, where $e = \#(\mathcal{P})$.

2. (a) Give an explicit example of a Δ^0_4 set which is neither Σ^0_3 nor Π^0_3.
   (b) Give an example of a set which cannot be classified in the arithmetical hierarchy.

3. Let A, B, C be recursively enumerable sets with $A = B \cup C$ and $B \cap C = \emptyset$. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the respective Turing degrees of A, B, C prove that $\mathbf{a} = \sup(\mathbf{b}, \mathbf{c})$.

   Note: The hardest part is to prove that $B \leq_T A$ and $C \leq_T A$. Your proof should use the assumption that A, B, C are r.e. sets. Without this assumption, the result would not be correct.

4. Construct an infinite descending sequence of Turing degrees
$$\mathbf{a}_0 > \mathbf{a}_1 > \cdots > \mathbf{a}_n > \mathbf{a}_{n+1} > \cdots$$
or prove that no such sequence exists.

5. Let $C(\sigma)$ denote the Kolmogorov complexity of a $0,1$-valued string σ. We have seen in class that
$$C(\sigma^\frown\tau) \ \leq\ 2C(\sigma) + 2C(\tau) + O(1)$$
for all $0,1$-valued strings σ and τ. Improve this inequality to
$$C(\sigma^\frown\tau) \ \leq\ C(\sigma) + 2\log_2 C(\sigma) + C(\tau) + O(1)$$
where $\log_2 x$ denotes the base 2 logarithm of x.

   Can you make further improvements?