Computability, Unsolvability, Randomness
Math 497A: Homework #6

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For each natural number n define

$$C_\varphi(n) \quad = \quad \mu e \left(\varphi_e^{(1)}(0) \simeq n \right) .$$

Intuitively, $C_\varphi(n)$ is the smallest "description" of n in terms of our standard enumeration of the 1-place partial recursive functions, $\varphi_e^{(1)}$, $e = 0, 1, 2, \ldots$. Note that C_φ is a total 1-place function, but it is not recursive.

Consider the set

$$S \quad = \quad \{ n \mid C_\varphi(n) < \log \log \log n \} .$$

Intuitively, S is the set of all n such that n has a (relatively) small "description." For example, the number

$$n = (10 \text{ to the } 10 \text{ to the } 10 \text{ to the } 10 \text{ to the } 1,000,000,000 \text{ power})$$

belongs to S because, although it is very large, it is also very easy to describe.

Prove that S is a simple set. This means:

1. S is recursively enumerable.

2. The complement of S is infinite.

3. The complement of S includes no infinite recursively enumerable set.