Computability, Unsolvability, Randomness
Math 497A: Homework #5

Stephen G. Simpson

Due Monday, October 1, 2007

1. Recall that a simple r.e. set is neither recursive nor many-one complete. Use Post's Theorem plus relativization to generalize this to higher levels of the arithmetical hierarchy.

   Conclude that for each $n \geq 1$ there exist Σ^0_n sets which are neither many-one complete (within the class of Σ^0_n sets) nor Δ^0_n.

2. (a) Given a Σ^0_1 predicate $P(x, y)$ such that $\forall x \, \exists y \, P(x, y)$ holds, prove that there exists a recursive function $f(x)$ such that $\forall x \, P(x, f(x))$ holds.

   (b) Use Post's Theorem plus relativization to generalize the previous result to higher levels of the arithmetical hierarchy.

   Conclude that for all $n \geq 1$, given a Σ^0_n predicate $P(x, y)$ such that $\forall x \, \exists y \, P(x, y)$ holds, there exists a Δ^0_n function $f(x)$ such that $\forall x \, P(x, f(x))$ holds.

3. Prove the following:

   (a) Every infinite recursively enumerable set includes an infinite recursive set.

   (b) Every infinite recursive set includes a recursively enumerable set which is not recursive.

   (c) Every infinite recursive set is the union of two disjoint infinite recursive sets.

   (d) Every infinite recursively enumerable set is the union of two disjoint infinite recursively enumerable sets.

   (e) (Extra Credit) Every recursively enumerable set which is nonrecursive is the union of two disjoint recursively enumerable sets which are nonrecursive.

4. Given a nonrecursive recursively enumerable set A, prove that we can find a simple set B such that $A \equiv_T B$.

   Hint: Use a deficiency set.

5. Prove the followng theorem:

   > Given a Turing degree $\mathbf{d} \geq \mathbf{0}'$, we can find a Turing degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{d}$.

   Thus, the range of the Turing jump operator consists precisely of the Turing degrees which are $\geq \mathbf{0}'$.

   Hint: Use the technique of finite approximation.