Computability, Unsolvability, Randomness
Math 497A: Homework #4

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Due Monday, September 24, 2007

In this set of exercises we explore the structure of the Turing degrees. Given two Turing degrees $\mathbf{a}$ and $\mathbf{b}$, we know that the least upper bound $\sup(\mathbf{a},\mathbf{b})$ always exists. Exercises 2, 4, and 7 below show that the greatest lower bound $\inf(\mathbf{a},\mathbf{b})$ sometimes exists and sometimes does not exist.

For any Turing oracle f we have

$$f' = H^f = \{x \mid \varphi_x^{(1),f}(0)\downarrow\} = \text{the Halting Problem relative to } f.$$

We know that f' is a complete Σ^0_1 set relative to the oracle f. For any Turing degree $\mathbf{a} = \deg_T(f)$ we define

$$\mathbf{a}' = \deg_T(f') = \text{the Turing jump of } \mathbf{a}.$$

Clearly $\mathbf{a} < \mathbf{a}'$ holds for all $\mathbf{a}$. Thus, starting with any Turing degree $\mathbf{a}$, we have an ascending sequence of Turing degrees

$$\mathbf{a} < \mathbf{a}' < \mathbf{a}'' < \cdots < \mathbf{a}^{(n)} < \mathbf{a}^{(n+1)} < \cdots$$

In particular, starting with the zero Turing degree $\mathbf{0}$, we have the ascending sequence

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(n)} < \mathbf{0}^{(n+1)} < \cdots$$

corresponding to the arithmetical hierarchy.

1. Given Turing oracles f and g, prove that the following conditions are pairwise equivalent:

   (a) $f \leq_T g$

   (b) $H^f \leq_m H^g$

   (c) all partial f-recursive functions are partial g-recursive

   (d) all total f-recursive functions are g-recursive.

2. Use finite approximations to construct Turing degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} > \mathbf{0}$ and $\mathbf{b} > \mathbf{0}$ and $\inf(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

3. Use finite approximations to construct Turing degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{0}'$ and $\mathbf{b} < \mathbf{0}'$ and $\sup(\mathbf{a}, \mathbf{b}) = \mathbf{0}'$.

4. Combine and generalize Exercises 2 and 3 to prove the following:

   Given two Turing degrees $\mathbf{c}, \mathbf{d}$ such that $\mathbf{c}' \leq \mathbf{d}$, we can find two Turing degrees $\mathbf{a}, \mathbf{b}$ such that $\inf(\mathbf{a}, \mathbf{b}) = \mathbf{c}$ and $\sup(\mathbf{a}, \mathbf{b}) = \mathbf{d}$.

5. Prove the following result.

   Given an ascending sequence of Turing degrees

   $$\mathbf{d}_0 < \mathbf{d}_1 < \cdots < \mathbf{d}_n < \mathbf{d}_{n+1} < \cdots$$

   we can find a pair of Turing degrees $\mathbf{a}, \mathbf{b}$ such that for all Turing degrees $\mathbf{c}$

   $$\exists n\, (\mathbf{c} \leq \mathbf{d}_n) \quad \text{if and only if} \quad \mathbf{c} \leq \mathbf{a} \text{ and } \mathbf{c} \leq \mathbf{b}\,.$$

6. Use the result of Exercise 5 to prove that no ascending sequence of Turing degrees has a least upper bound.

7. For any pair of Turing degrees $\mathbf{a}, \mathbf{b}$ as in Exercise 5, prove that the greatest lower bound $\inf(\mathbf{a}, \mathbf{b})$ does not exist.