Computability, Unsolvability, Randomness
Math 497A: Homework #3

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Recall that $W_x = \mathrm{dom}(\varphi_x^{(1)})$. Note that W_x, $x = 0, 1, 2, \ldots$, is the standard recursive enumeration of the recursively enumerable subsets of $\mathbb{N}$.

1. Which many-one reducibility relations hold or do not hold among the following sets and their complements?

   $$K = \{x \mid x \in W_x\}$$
   $$H = \{x \mid 0 \in W_x\}$$
   $$T = \{x \mid W_x = \mathbb{N}\}$$
   $$E = \{x \mid W_x = \emptyset\}$$
   $$S = \{x \mid W_x \text{ is infinite}\}$$

   Prove your answers.

   Hint: Each of these sets is many-one complete within an appropriate level of the arithmetical hierarchy.

2. A set $P \subseteq \mathbb{N}$ is said to be *productive* if there exists a total recursive function $h(x)$ such that for all x, if $W_x \subseteq P$ then $h(x) \notin W_x$ and $h(x) \in P$. Such a function is called a *productive function* for P.

   A *creative set* is a recursively enumerable set whose complement is productive.

   Prove the following.

   (a) K is creative.

   (b) If A and B are recursively enumerable sets and $A \leq_m B$ and A is creative, then B is creative.

   (c) If B is recursively enumerable and many-one complete, then B is creative.

(d) (Extra Credit) If B is creative, then B is many-one complete.

(e) (Extra Credit) If A and B are creative, then A and B are *recursively isomorphic*. This means that there exists a recursive permutation of $\mathbb{N}$, call it g, such that $x \in A$ if and only if $g(x) \in B$, for all x.

3. A set $I \subseteq \mathbb{N}$ is said to be *immune* if I is infinite yet includes no infinite recursively enumerable set.

   A *simple set* is a recursively enumerable set whose complement is immune.

   Prove the following.

   (a) If A is simple, then A is not recursive.

   (b) If A is simple, then A is not creative.

4. Let $f : \mathbb{N} \xrightarrow{1-1} \mathbb{N}$ be a one-to-one total recursive function such that the range of f is nonrecursive. The *deficiency set* of f is defined as

$$D_f = \{x \mid \exists y \, (x < y \wedge f(x) > f(y))\} .$$

   Prove that D_f is a simple set.

   Conclude that there exist recursively enumerable sets which are neither recursive nor many-one complete.

5. (Extra Credit) Generalize Exercises 2, 3, 4 to higher levels of the arithmetical hierarchy. Conclude that for each $n \geq 1$ there exist Σ^0_n sets which are neither Δ^0_n nor many-one complete within the class of Σ^0_n sets.