Computability, Unsolvability, Randomness
Math 497A: Homework #2

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1. Let r be a positive real number. Prove that r is computable if and only if the number-theoretic function

$$f(n) = \text{the } n\text{th decimal digit of } r$$

   is computable.

2. Consider the 2-place computable number-theoretic function $f(x, y) = x + y$. Exhibit three different indices of f.

   (By an *index* of a partial recursive function, we mean the Gödel number of some program which computes the function.)

3. If f is a computable permutation of $\mathbb{N}$, prove that the inverse permutation f^{-1} is also computable.

   (Here $f^{-1}(y) = x$ if and only if $f(x) = y$. By a *computable permutation of* $\mathbb{N}$ we mean a computable 1-place function $f : \mathbb{N} \to \mathbb{N}$ which maps $\mathbb{N}$ one-to-one onto $\mathbb{N}$.)

4. Generalize the previous exercise as follows. Prove that if ψ is a 1-place partial recursive function which is one-to-one, then the inverse function ψ^{-1} is again partial recursive.

5. Consider the sets

$$K_n = \{x \in \mathbb{N} \mid \varphi_x^{(1)}(x) \simeq n\}$$

   where $n = 0, 1, 2, \ldots$. Show that the sets K_0 and K_1 are recursively inseparable. More generally, show that K_m and K_n are recursively inseparable for all m, n such that $m \neq n$.

   (Two sets $A, B \subseteq \mathbb{N}$ are said to be *recursively separable* if there exists a recursive function $f : \mathbb{N} \to \{0, 1\}$ such that $f(n) = 1$ for all $n \in A$, and

$f(n) = 0$ for all $n \in B$. Otherwise, A and B are said to be *recursively inseparable*.)

6. Let $\psi(x)$ and $\theta(x)$ be 1-place partial recursive functions. We say that *ψ is reducible to θ* if there exists a 1-place total recursive function $h(x)$ such that $\psi(x) \simeq \theta(h(x))$ for all $x \in \mathbb{N}$. We refer to $h(x)$ as a *reduction function*, and we say that *h reduces ψ to θ*. We say that θ is *universal* if all 1-place partial recursive functions are reducible to θ.

   (a) Prove that the 1-place partial recursive function $\varphi_x^{(1)}(x)$ is universal.

   (b) Give some additional examples of 1-place partial recursive functions which are universal.

   (c) Prove that if θ is universal then the domain of θ is not recursive. (The *domain of θ* is defined to be the set $\mathrm{dom}(\theta) = \{x \mid \theta(x) \downarrow\}$.)

   (d) Construct a 1-place partial recursive function θ which is universal via linear reduction functions.

   (This means that each 1-place partial recursive function is reducible to θ by means of a reduction function which is linear. We say that $h(x)$ is *linear* if there exist constants a and b such that $h(x) = ax + b$ for all x.)

7. (Extra Credit). Prove that any two universal partial recursive functions θ_1 and θ_2 are *recursively isomorphic*. This means that there exists a computable permutation of $\mathbb{N}$, call it f, such that

$$\theta_1(x) \simeq y \quad \text{if and only if} \quad \theta_2(f(x)) \simeq f(y)$$

   for all x and y.