Computability, Unsolvability, Randomness
Math 497A: Homework #1

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1. Exhibit a register machine program which computes the exponential function, $\exp(x, y) = x^y$. Remember that the variables x and y range over $\mathbb{N}$, the set of natural numbers. Note that $x^0 = 1$ for all x, even for $x = 0$.

2. Exhibit a register machine program which computes the function

   $$\mathrm{Rem}(y, x) = \text{the remainder of } y \text{ on division by } x.$$

   For example, Rem(17,5)=2.

3. Assume that $P(x, y, z)$ is a 3-place predicate which is computable. Consider the 2-place partial function $\psi(x, y)$ defined as follows:

   > $\psi(x, y) \simeq$ the least z such that $P(x, y, z)$ holds, if such a z exists. If such a z does not exist, then $\psi(x, y)$ is undefined.

   Use register machine programs to prove that the function ψ is computable. (More precisely, ψ is partial recursive.)

   Hint: Since P is a computable predicate, we may assume that we have a program $\mathcal{P}$ which computes the characteristic function of P. Show how to embed $\mathcal{P}$ (or perhaps a variant of $\mathcal{P}$) into a larger program, call it $\mathcal{Q}$, such that $\mathcal{Q}$ computes ψ. The idea of $\mathcal{Q}$ is that, given x and y, $\mathcal{Q}(x, y)$ searches sequentially through the integers $z = 0$, $z = 1$, $z = 2$, ..., to find the first z such that $P(x, y, z)$ holds. In particular, $\mathcal{Q}$ will have the property that, for all x and y, $\mathcal{Q}(x, y)$ eventually halts if and only if $P(x, y, z)$ holds for some z.

4. Use the results of Problems 2 and 3 to prove that the 2-place function $\mathrm{LCM}(x, y)$ = the least common multiple of x and y is computable. Deduce that the 2-place function $\mathrm{GCD}(x, y)$ = the greatest common divisor of x and y, is also computable.

5. (Approximating the square root of 2.) Consider successive rational approximations of $\sqrt{2}$ given by Newton's method:

$$x_0 = 1, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f(x) = x^2 - 2$. The first few values are $x_0 = 1$, $x_1 = 3/2$, $x_2 = 17/12$, $x_3 = 577/408$. Let $a(n)$ and $b(n)$ respectively be the numerator and denominator of x_n. Thus $a(n)$ and $b(n)$ are 1-place number-theoretic functions. The first few values are $a(0) = b(0) = 1$, $a(1) = 3$, $b(1) = 2$, $a(3) = 17$, $b(3) = 12$, $a(3) = 577$, $b(3) = 408$. Use primitive recursion to prove that the functions $a(n)$ and $b(n)$ are computable.

6. A positive real number r is said to be *computable* if there exist computable sequences of positive integers a_n, b_n, $n = 0, 1, 2, \ldots$, such that

$$r = \lim_{n\to\infty} \frac{a_n}{b_n}$$

and in addition

$$\left| r - \frac{a_n}{b_n} \right| < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Give a convincing argument that all of the standard examples of positive real numbers including $\sqrt{2} = 1.41421\cdots$, $e = 2.71828\cdots$, $\pi = 3.14159\cdots$, etc., are computable.

7. Prove that the sum, product, and quotient of two computable positive real numbers are computable.