Math 485, Graph Theory: Midterm Exam
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Solutions

1. (a) If G is a k-regular graph with n vertices, how many edges does G have?

   Solution. $kn/2$, by the degree-sum formula.

   (b) Draw a 3-regular graph with 11 vertices or prove that no such graph exists.

   Solution. Part (a) implies that no such graph exists, because in this case $kn/2 = 3 \cdot 11/2$ which is not an integer.

2. Recall that Q_3 is the 3-cube. How many automorphisms does Q_3 have? Justify your answer.

   Solution. The answer is 48. To see this, pick a vertex v of the 3-cube, and let e_1, e_2, e_3 be the edges which are incident to v. An automorphism must carry v to some vertex v'. There are 8 choices for v'. Once v has been chosen, there are $3! = 6$ choices for mapping e_1, e_2, e_3 to the edges e'_1, e'_2, e'_3 which are incident to v'. Once these $8 \cdot 6 = 48$ choices have been made, the automorphism is completely determined.

3. True or false.

   (a) The number of spanning trees in the n-cycle is n. **True.**

   (b) Dijkstra's algorithm applied to a connected weighted graph produces a minimum-weight spanning tree. **False.**

   (c) Every connected bipartite graph contains a cycle of even length. **False.** For instance, $K_{1,1}$ or $K_{1,2}$.

   (d) The Petersen graph has 25 edges. **False.**

   (e) A weakly connected digraph with $\deg^+(v) = \deg^-(v)$ for all vertices v is strongly connected.
   True. (This is implicit in Euler's Theorem for digraphs.)

4. It is known that K_4 has 16 spanning trees. Prove this using the Matrix Tree Theorem.

   Solution. The Matrix Tree Theorem tells us that

$$\tau(K_4) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16.$$

5. Let G be an acyclic digraph with vertices u, v, w consisting of k pairwise disjoint uv-paths plus l pairwise disjoint vw-paths. Consider the two-person positional game starting at u. Under what conditions does the first player have a winning strategy?

   Solution. The first player has a winning strategy under either of the following circumstances:

   (a) At least one of the uv-paths is even, and at least one of the vw-paths is odd.

   (b) At least one of the uv-paths is odd, and all of the vw-paths are even.

   Otherwise, the second player has a winning strategy.

6. It is known that K_n has n^{n-2} spanning trees. If e is an edge of K_n, how many spanning trees does $K_n - e$ have? Justify your answer.

   Solution. We reason probabilistically. Let e be a randomly chosen edge of K_n, and let T be a randomly chosen spanning tree of K_n. The probability that e belongs to T is the number of edges in T divided by the total number of edges. In other words, $(n-1)/(n \text{ choose } 2)$, which simplifies to $2/n$. Therefore, the probability that e does not belong to T is $1 - 2/n$. Therefore, the number of spanning trees not containing e is $n^{n-2}(1-2/n)$ which simplifies to $n^{n-3}(n-2)$.

7. The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... where each number is obtained by adding the previous two numbers. Let ϕ_n be the nth Fibonacci number. Thus $\phi_1 = 1$, $\phi_2 = 1$, $\phi_3 = 2$, $\phi_4 = 3$, $\phi_5 = 5$, $\phi_6 = 8$, $\phi_7 = 13$, etc.

   (a) Write a recursion formula for ϕ_n.

   (b) Solve your recursion to obtain an explicit formula for ϕ_n.

   (c) Find $\lim_{n\to\infty} \dfrac{\phi_{n+1}}{\phi_n}$.

   Solution.

   (a) The recursion
   $$\phi_n = \phi_{n-1} + \phi_{n-2}$$
   holds for all $n \geq 3$. It also holds for $n = 2$ if we define $\phi_0 = 0$.

   (b) The characteristic equation is $\alpha^2 = \alpha + 1$. The roots of this are $\alpha = (1 \pm \sqrt{5})/2$, so the general solution is
   $$\phi_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using the initial conditions $\phi_0 = 0$ and $\phi_1 = 1$, we can solve for A and B to get $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$. Thus

$$\phi_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and this is our explicit formula.

(c) Since $\dfrac{1+\sqrt{5}}{2} > 1 > \left|\dfrac{1-\sqrt{5}}{2}\right|$, the term involving $\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ is negligible for large n. Thus $\phi_n \approx \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ and in particular $\displaystyle\lim_{n\to\infty} \frac{\phi_{n+1}}{\phi_n} = \frac{1+\sqrt{5}}{2}$.

8. Consider a weighted bipartite graph with weights as indicated:

	y_1	y_2	y_3	y_4
x_1	4	3	3	3
x_2	4	2	2	2
x_3	4	1	1	1
x_4	4	1	3	4

Use the Hungarian algorithm to find a matching of maximum weight and a weighted covering of minimum cost. Show all of the intermediate steps, including weighted coverings and the equality subgraphs which are associated with them.

Solution. We omit the steps of the Hungarian algorithm. There are several maximum-weight matchings. One of them is $M = \{x_1y_3, x_2y_2, x_3y_1, x_4y_4\}$ for a total weight of $3+2+4+4 = 13$. A minimum-cost covering is given by $u_1 = 2, u_2 = u_3 = 1, u_4 = 2, v_1 = 3, v_2 = v_3 = 1, v_4 = 2$ for a total cost of $2+1+1+2+3+1+1+2 = 13$.