Math 485, Graph Theory: Midterm Exam

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8 problems

1. (a) If G is a k-regular graph with n vertices, how many edges does G have?

   (b) Draw a 3-regular graph with 11 vertices or prove that no such graph exists.

2. Recall that Q_3 is the 3-cube. How many automorphisms does Q_3 have? Justify your answer.

3. True or false.

   (a) The number of spanning trees in the n-cycle is n.

   (b) Dijkstra's algorithm applied to a connected weighted graph produces a minimum-weight spanning tree.

   (c) Every connected bipartite graph contains a cycle of even length.

   (d) The Petersen graph has 25 edges.

   (e) A weakly connected digraph with $\deg^+(v) = \deg^-(v)$ for all vertices v is strongly connected.

4. It is known that K_4 has 16 spanning trees. Prove this using the Matrix Tree Theorem.

5. Let G be an acyclic digraph with vertices u, v, w consisting of k pairwise disjoint uv-paths plus l pairwise disjoint vw-paths. Consider the 2-person positional game starting at u. Under what conditions does the first player have a winning strategy?

6. It is known that K_n has n^{n-2} spanning trees. If e is an edge of K_n, how many spanning trees does $K_n - e$ have? Justify your answer.

7. The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... where each number is obtained by adding the previous two numbers. Let ϕ_n be the nth Fibonacci number. Thus $\phi_1 = 1$, $\phi_2 = 1$, $\phi_3 = 2$, $\phi_4 = 3$, $\phi_5 = 5$, $\phi_6 = 8$, $\phi_7 = 13$, etc.

   (a) Write a recursion formula for ϕ_n.

   (b) Solve your recursion to obtain an explicit formula for ϕ_n.

   (c) Find $\lim_{n\to\infty} \dfrac{\phi_{n+1}}{\phi_n}$.

8. Consider a weighted bipartite graph with weights as indicated:

	y_1	y_2	y_3	y_4
x_1	4	3	3	3
x_2	4	2	2	2
x_3	4	1	1	1
x_4	4	1	3	4

   Use the Hungarian algorithm to find a matching of maximum weight and a weighted covering of minimum cost. Show all of the intermediate steps, including weighted coverings and the equality subgraphs which are associated with them.