# Math 485, Graph Theory: Final Exam

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13 problems, 220 points

1. (15 points) Draw a simple connected planar graph which is 6-regular, or prove that no such graph exists.

2. (15 points) Exhibit a planar graph $G$ which has at least two different (non-isomorphic) dual graphs – depending on the planar representation of $G$. Exhibit two planar representations of $G$ with different dual graphs. Make your $G$ as small as possible.

3. (15 points) Draw a connected graph $G$ consisting of three blocks which are isomorphic to $K_5$ and $K_{3,3}$ and $C_{10}$ respectively. How many spanning trees does $G$ have?

4. (15 points) State the Graph Minor Theorem. Use it to prove that graphs of genus $\leq g$ can be characterized by a finite list of forbidden minors.

5. (20 points) Recall that the octahedron is a simple connected 4-regular planar graph whose dual is simple and 3-regular. Recall that, according to Fáry's Theorem, any simple planar graph has a straight-line representation.

   (a) Draw a straight-line representation of the octahedron.

   (b) Write a determinant for the number of spanning trees of the octahedron.

6. (20 points) True or False.

   (a) $\tau(G) \geq \chi(G)$ for any graph $G$.

   (b) Any graph of genus $\leq 3$ is 9-colorable.

   (c) A simple graph of genus $\leq 3$ with $n$ vertices has $\leq 3n + 12$ edges.

   (d) For $k > 0$, any $k$-regular simple graph has a perfect matching.

   (e) In a tree, every edge is a cut edge.

7. (20 points) Use Heawood's formula to find a lower bound for the genus of $K_{12}$, the complete graph on 12 vertices.

8. (15 points) Recall that $L_n$ is the $n$-ladder, with $2n$ vertices. What is the diameter of $L_n$? What is the chromatic number of $L_n$? What is the genus of $L_n$? How many spanning trees does $L_n$ have?

9. (20 points) Let $G$ be a planar bipartite graph. Prove that the dual graph $G^*$ is Eulerian.

10. (15 points) Draw a graph $G$ such that the number of spanning trees of $G$ is given by the determinant

$$\begin{vmatrix} 3 & -2 & 0 & -1 \\ -2 & 6 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ -1 & -1 & 0 & 4 \end{vmatrix}$$

11. (15 points) Let $G$ be a graph with $n$ vertices. Recall that $\tau(G)$ is the number of spanning trees of $G$. Let $G_k$ be the graph obtained from $G$ upon replacing each edge by $k$ parallel edges. What is the relationship between $\tau(G)$ and $\tau(G_k)$? Justify your answer.

12. (15 points) Let $G$ be a connected graph with exactly $n$ vertices and $m$ edges. If $G$ contains exactly two cycles, what can you say about the relationship between $n$ and $m$? Justify your answer.

13. (20 points) Consider a weighted bipartite graph where the weights are given by the following matrix.

$$\begin{matrix} 4 & 2 & 1 \\ 5 & 2 & 1 \\ 6 & 7 & 1 \end{matrix}$$

   (a) By inspection, find a matching $M$ of maximum weight. (You are not required to use the Hungarian algorithm.)

   (b) Find a weighted vertex covering $C$ of minimum weight.

   (c) What relationship must hold between $M$ and $C$?