Mathematical Logic and Foundations of Mathematics

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Open House / Graduate Conference

April 2–3, 2010

Foundations of mathematics (f.o.m.) is the study of the most basic concepts and logical structure of mathematics as a whole.

Among the most basic mathematical concepts are: number, shape, set, function, algorithm, mathematical proof, mathematical definition, mathematical axiom, mathematical theorem.

Some typical questions in f.o.m. are:

1. What is a number?
2. What is a shape?

$\vdots$

6. What is a mathematical proof?

$\vdots$

10. What are the appropriate axioms for mathematics?

Mathematical logic gives some mathematically rigorous answers to some of these questions.

The concepts of "mathematical theorem" and "mathematical proof" are greatly clarified by the predicate calculus.

Actually, the predicate calculus applies to non-mathematical subjects as well.

Let Lxy be a 2-place predicate meaning

"x loves y".

We can express properties of loving as sentences of the predicate calculus.

$\forall x \, \exists y \, Lxy$

$\exists x \, \forall y \, Lyx$

$\forall x \, (Lxx \Rightarrow \neg \, \exists y \, Lyx)$

$\forall x \, \forall y \, \forall z \, (((\neg \, Lyx) \wedge (\neg \, Lzy)) \Rightarrow Lzx)$

$\forall x \, ((\exists y \, Lxy) \Rightarrow Lxx)$

There is a deterministic algorithm
(the Tableau Method) which shows us
(after a finite number of steps)
that particular sentences are logically valid.
For instance, the tableau

$$\exists x\,(Sx \wedge \forall y\,(Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy)))$$

$$Sa \wedge \forall y\,(Eya \Leftrightarrow Sy \wedge \neg\, Eyy)$$

$$Sa$$

$$\forall y\,(Eya \Leftrightarrow Sy \wedge \neg\, Eyy)$$

$$Eaa \Leftrightarrow (Sa \wedge \neg\, Eaa)$$

$$\begin{array}{cc}
/ & \backslash \\
Eaa & \neg\, Eaa \\
Sa \wedge \neg\, Eaa & \neg\,(Sa \wedge \neg\, Eaa) \\
Sa & \begin{array}{cc} / & \backslash \\ \neg\, Sa & \neg\,\neg\, Eaa \\ & Eaa \end{array} \\
\neg\, Eaa &
\end{array}$$

tells us that the sentence

$$\neg\,\exists x\,(Sx \wedge \forall y\,(Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy)))$$

is logically valid. This is the Russell Paradox.

Two significant results in mathematical logic: (Gödel, Tarski, . . .)

1. The theory of the natural numbers is *undecidable*. This means, there is no algorithm for determining whether a given sentence of the predicate calculus is true in the structure $(\mathbb{N}, +, \times, =)$.

2. The theory of the real numbers is *decidable*. This means, there is an algorithm for determining whether a given sentence of the predicate calculus is true in the structure $(\mathbb{R}, +, \times, =)$.

Mathematical logic has many connections with other fields outside mathematics: computer science, philosophy, economics, ...

Some exciting current research topics in mathematical logic are:

- Reverse Mathematics. What are the weakest possible axioms needed to prove particular mathematical theorems?

- Algorithmic randomness. What do we mean by a "random" point in a probability space? (E.g., $[0,1]$ with the Lebesgue probability measure.) Computation using coin flips.

- Complexity of classification problems. For instance, how hard is it to classify compact metric spaces up to isometry?

- Axiomatic set theory. What are the appropriate axioms for mathematics as a whole?

Two introductory graduate courses:
(**no prerequisites**)

1. Math 557, **Mathematical Logic**.

   - propositional calculus
   - predicate calculus
   - proof systems
   - theories and models
   - the Incompleteness Theorems of Gödel

2. Math 558, **Foundations of Mathematics**.

   - computability and unsolvability
   - undecidability of $(\mathbb{N}, +, \times, =)$
   - decidability of $(\mathbb{R}, +, \times, =)$
   - introduction to set theory
   - axiomatic set theory
   - the Axiom of Choice
   - the Continuum Hypothesis