MATH 457: Introduction to Mathematical Logic

Homework #6

Due April 9, 2012

1. Write a register machine program which computes the total function $f(m,n) = m^n$. (Recall that $m^0 = 1$ for all m including $m = 0$.)

2. Write register machine programs which compute the following functions.

   (a) $\psi_1(m,n) = \begin{cases} q \text{ if } m = qn \text{ where } q \in \mathbb{N} \text{ and } 0 < n, \\ \text{undefined otherwise.} \end{cases}$

   (b) $\psi_2(m,n) = \begin{cases} q \text{ if } m = qn \text{ where } q \in \mathbb{N} \text{ and } 0 < n, \\ 0 \text{ otherwise.} \end{cases}$

   (c) $\psi_3(m,n) = \begin{cases} q \text{ if } m = qn + r \text{ where } q, r \in \mathbb{N} \text{ and } 0 \leq r < n, \\ \text{undefined if } n = 0. \end{cases}$

   Note: Roughly speaking, for all $m, n \in \mathbb{N}$ we have $\psi_1(m,n) = \psi_2(m,n) = \psi_3(m,n) =$ the quotient of m on division by n. However, there are significant differences among these three functions.

3. Define
$$h(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

   Using primitive recursion, define $f(m,i) = h^i(m)$ as follows:
$$\begin{aligned} f(m,0) &= m, \\ f(m,i+1) &= h(f(m,i)). \end{aligned}$$

   Finally, define $\psi(m) =$ the least i such that $h^i(m) = 1$.

   Write a register machine program which computes ψ.

   Note: It is unknown whether $\psi(m)$ is defined for all $m \geq 1$. This is a famous open problem in mathematics.