Math 312, Intro. to Real Analysis: Final Exam: Solutions

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1. True or false (3 points each).

   (a) For all sequences of real numbers (s_n) we have $\liminf s_n \leq \limsup s_n$. **True.**

   (b) Every bounded sequence of real numbers has at least one subsequential limit. **True.**

   (c) If the functions f_n are continuous on $[0,1]$ and converge uniformly to the function f on $[0,1]$, then f is uniformly continuous on $[0,1]$. **True.**

   (d) If the radius of convergence of a power series $\sum a_k x^k$ is R, and if $0 < R < \infty$, then the series $\sum a_k x^k$ converges uniformly on $(-R, R)$. **False.**

   (e) The integral of the limit is equal to the limit of the integrals. **False.**

2. (10 points) The real number system $\mathbb{R}$ has been characterized in terms of Axioms A1–A4, M1–M4, DL, O1–O5, and the Completeness Axiom. Which of these axioms fail for the rational number system $\mathbb{Q}$? Give one or more examples illustrating your answers.

   Solution. The only axiom that fails for $\mathbb{Q}$ is the Completeness Axiom. For example, the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded but has no least upper bound in $\mathbb{Q}$.

3. (10 points)

   (a) State the formal definition of what it means for a sequence of real numbers (s_n) to converge to a limit s.

   Solution. Given $\epsilon > 0$ we can find N such that $|s_n - s| < \epsilon$ for all $n > N$.

   (b) In terms of your definition from part (a), prove directly that

$$\lim_{n\to\infty} \frac{\pi}{\sqrt{n - 100\pi}} = 0.$$

Solution. Clearly $|\pi/\sqrt{n} - 100\pi| < \epsilon$ whenever $n > 100\pi + \pi^2/\epsilon^2$. Thus we may take $N = 100\pi + \pi^2/\epsilon^2$.

4. (7 points) Calculate $\lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \cdots + (-1)^n \frac{3}{2^n} \right)$.

   Solution. The sum of this geometrical series is

$$\frac{3}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{4} - \cdots\right) = \frac{3}{2} \cdot \frac{1}{1 - (-\frac{1}{2})} = 1.$$

5. (a) (2 points) Give an example of a bounded sequence of real numbers with exactly two subsequential limits.

   Solution. $0, 1, 0, 1, 0, 1, \ldots$.

   (b) (2 points) Give an example of a bounded sequence of real numbers with exactly five subsequential limits.

   Solution. $0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \ldots$.

   (c) (4 points) Give an example of a bounded sequence of real numbers with infinitely many subsequential limits.

   Solution. $0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots$.

6. (10 points) Give an example of a sequence of continuous functions on $[0, 1]$ such that $f_n \to 0$ pointwise but not uniformly on $[0, 1]$.

   Solution. Let $f_n(0) = 0$, $f_n(1/n) = 1$, $f_n(2/n) = f_n(1) = 0$, and let $f_n(x)$ be linear on the intervals $[0, 1/n]$, $[1/n, 2/n]$ and $[2/n, 1]$.

7. (15 points) Prove that if $\sum |a_k|$ is convergent then $\sum a_k$ is convergent.

   Hint: Use the Triangle Inequality.

   Solution. We know that $\sum a_k$ is convergent if and only if the Cauchy criterion holds: given $\epsilon > 0$ we can find N such that $|a_m + a_{m+1} + \cdots + a_n| < \epsilon$ whenever $n > m > N$. The Triangle Inequality tells us that

$$|a_m + a_{m+1} + \cdots + a_n| \leq |a_m| + |a_{m+1}| + \cdots + |a_n|.$$

   Thus, $\sum a_k$ satisfies the Cauchy criterion whenever $\sum |a_k|$ does.

8. (3 points each) For each of the following series, tell whether the series is convergent or divergent. State which convergence/divergence test you are using, and show any needed calculations.

   (a) $\sum \frac{1}{2^n}$

   Solution. Convergent geometric series.

(b) $\sum \frac{(-1)^n}{\sqrt{n}}$

Solution. Convergent by the Alternating Series Test.

(c) $\sum (-1)^n \frac{n}{100n + 1000}$

Solution. Divergent by the nth term test, because $\frac{n}{100n+1000} \to \frac{1}{100} \neq 0$.

(d) $\sum \left(\frac{7n}{8n+1}\right)^n$

Solution. Convergent by the Root Test, because $\frac{7n}{8n+1} \to \frac{7}{8} < 1$.

(e) $\sum \frac{1}{n \log n}$

Solution. Divergent by the Integral Test.

9. (15 points) For each of the following functions, say whether the function is *continuous* and/or *uniformly continuous* on each of the three intervals

$$[0, 1], \qquad (0, 1), \qquad (2, \infty).$$

You are not required to justify or prove your answers.

(a) $\sin x$

Solution. Uniformly continuous on $(-\infty, \infty)$.

(b) e^x

Solution. Uniformly continuous on any bounded interval; continuous but not uniformly continuous on $[a, \infty)$ for any a.

(c) $|x - \frac{1}{2}| + |x - 3|$

Solution. Uniformly continuous on $(-\infty, \infty)$.

(d) $\frac{1}{1-x}$

Solution. Undefined at $x = 1$, hence not continuous on $[0, 1]$. Continuous but not uniformly continuous on $(0, 1)$. Uniformly continuous on $[2, \infty)$.

(e) $\sum_{n=0}^{\infty} x^n$

Solution. Undefined at $x = 1$, hence not continuous on $[0, 1]$. Continuous but not uniformly continuous on $(0, 1)$. Undefined (divergent) for $x > 1$.

10. (10 points)

    (a) Prove directly that $x^n \to 0$ uniformly on the interval $[-0.99, 0.99]$.

    Solution. For all $x \in [-0.99, 0.99]$ we have $|x^n| \leq 0.99^n$, and $0.99^n \to 0$ independently of x.

    (b) Does $x^n \to 0$ uniformly on the interval $(-1, 1)$? Justify your answer.

    Solution. No, because for any n we have $0^n = 0$ and

    $1^n = 1$, hence we can find x such that $0 < x < 1$ and $x^n = 1/2$.

11. (10 points)

    (a) Calculate $\displaystyle\lim_{n\to\infty} \int_{-1}^{1} \sin^n x \, dx$.

    Solution. We have

    $$\lim_{n\to\infty} \int_{-1}^{1} \sin^n x \, dx = \int_{-1}^{1} \lim_{n\to\infty} \sin^n x \, dx = \int_{-1}^{1} 0 \, dx = 0.$$

    (b) Justify your calculation for part (a) by stating an appropriate property of the functions $\sin^n x$ and an applicable theorem.

    Solution. We have $\sin^n x \to 0$ uniformly on $[-1, 1]$. The applicable theorem says: if $f_n \to f$ uniformly on $[a, b]$ and f_n and f are continuous, then $\lim \int_a^b f_n = \int_a^b f$.

12. (15 points) For each of the following series, determine the set of all x such that the series converges at x. Show any needed calculations.

    (a) $\displaystyle\sum_{n=0}^{\infty} 2\cos^n x$

    Solution. Diverges whenever $\cos x = \pm 1$, i.e., $x = \pm n\pi$. Converges everywhere else, because then $|\cos x| < 1$.

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{nx^n}$

    Solution. Diverges for $-1 < x \leq 1$, converges everywhere else.

    (c) $\displaystyle\sum_{n=0}^{\infty} (2 + \sqrt{n})x^n$

    Solution. Converges for $-1 < x < 1$, diverges everywhere else.

    (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(10x)^n}{n^2 + 1}$

    Solution. Converges for $-1/10 \leq x \leq 1/10$, diverges everywhere else.

(e) $\sum_{n=0}^{\infty} n(x-2)^n$

Solution. Converges for $1 < x < 3$, diverges everywhere else.

13. (10 points)

(a) Determine the coefficients a_k for $k = 0, 1, 2, \ldots$ such that

$$\sum_{k=0}^{\infty} a_k x^k = \frac{x^2+1}{x-1}\,. \tag{1}$$

Solution. We have

$$\frac{x^2+1}{x-1} = \frac{(x+1)(x-1)+2}{x-1} = 1+x-\frac{2}{1-x} = 1+x-2(1+x+x^2+\cdots).$$

Thus $a_0 = a_1 = -1$, and $a_k = -2$ for all $k \geq 2$.

(b) Over what interval is the above equation (1) valid?

Solution. The equation is valid over the interval $(-1, 1)$.