Math 312, Intro. to Real Analysis: Homework #7 Solutions

Stephen G. Simpson

Wednesday, April 29, 2009

The assignment consists of Exercises 20.1, 20.18, 23.1, 23.4, 23.6, 24.1, 24.2, 24.6, 24.14, 24.17, 25.3, 25.6, 26.2, 26.6, 26.7 in the Ross textbook. Each exercise counts 5 points.

20.1. For $f(x) = x/|x| = \operatorname{sgn} x$ we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(x) = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f(x) = -1.$$

20.18. For $f(x) = \dfrac{\sqrt{1+3x^2}-1}{x^2}$ and $x \neq 0$ we have

$$f(x) = \frac{(1+3x^2)-1}{x^2(\sqrt{1+3x^2}+1)} = \frac{3}{\sqrt{1+3x^2}+1}$$

hence

$$\lim_{x\to 0} f(x) = \frac{3}{2}.$$

23.1. (a) $R = 1$, $I = (-1, 1)$.

(b) $R = \infty$, $I = (-\infty, \infty)$.

(c) $R = 1/2$, $I = [-1/2, 1/2]$.

(d) $R = 3$, $I = (-3, 3)$.

(e) $R = \infty$, $I = (-\infty, \infty)$.

(f) $R = 2$, $I = [-2, 2]$.

(g) $R = 4/3$, $I = [-4/3, 4/3)$.

(h) $R = 4$, $I = [-4, 4]$.

23.4. Note that $a_n = (6/5)^n$ if n is even, $(2/5)^n$ if n is odd.

Moreover $a_{n+1}/a_n = 2/(5\cdot 3^n)$ if n is even, $(2\cdot 3^n)/5$ if n is odd.

(a) $\limsup a_n^{1/n} = 6/5$, $\liminf a_n^{1/n} = 2/5$,

$\limsup a_{n+1}/a_n = \infty$, $\liminf a_{n+1}/a_n = 0$.

(b) $\sum a_n$ and $\sum(-1)^n a_n$ diverge, because $a_n \not\to 0$.

(c) For the power series $\sum a_n x^n$ we have $R = 5/6$ and $I = (-5/6, 5/6)$.

23.6. Suppose $a_n \geq 0$ and $x \geq 0$. If $\sum a_n x^n$ converges, then $\sum a_n(-x)^n$ converges absolutely (because $|a_n(-x)^n| = a_n x^n$), hence $\sum a_n(-x)^n$ converges in view of Corollary 14.7.

24.1. For $f_n(x) = \dfrac{1 + 2\cos^2 nx}{\sqrt{n}}$ we have $|\cos^2 nx| \leq 1$, hence $|f_n(x)| \leq \dfrac{3}{\sqrt{n}}$, hence $|f_n(x)| < \epsilon$ whenever $n > 9/\epsilon^2$. Since $9/\epsilon^2$ is independent of x, we see that f_n converges uniformly to 0.

24.2. For $f_n(x) = x/n$ we have:

(a) $f(x) = \lim f_n(x) = \lim \dfrac{x}{n} = 0$ (pointwise convergence).

(b) $f_n \to 0$ uniformly on $[0, 1]$. This is because for all $x \in [0, 1]$ we have

$$|f_n(x)| = \frac{|x|}{n} \leq \frac{1}{n} < \epsilon$$

whenever $n > 1/\epsilon$.

(c) $f_n \not\to 0$ uniformly on $[0, \infty)$. This is because for any n we can find $x \in [0, \infty)$ such that $|f_n(x)| \geq 1$. (An example of such an x is $x = n$.) Thus the definition of uniform convergence fails for $\epsilon = 1$.

24.6. Let $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ and $f(x) = x^2$.

(a) Clearly for all x we have $f_n(x) \to f(x)$, i.e., pointwise convergence.

(b) For $x \in [0, 1]$ we have

$$|f_n(x) - f(x)| = \left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|-\frac{2x}{n} + \frac{1}{n^2}\right| \leq \frac{2}{n} + \frac{1}{n^2} \to 0$$

independently of $x \in [0, 1]$. Thus $f_n \to f$ uniformly on $[0, 1]$.

24.14. Let $f_n(x) = \dfrac{nx}{1 + n^2x^2}$.

(a) For $x = 0$ we have $f_n(x) = 0$ for all n, hence $\lim f_n(0) = 0$. For $x \neq 0$ we have

$$f_n(x) = \frac{1}{\frac{1}{nx} + nx} \to 0$$

so again $\lim f_n(x) = 0$. Thus $f_n \to 0$ pointwise for all x.

(b) For all n we have $f_n(1/n) = 1/2$. Therefore, since $1/n \in [0, 1]$, we see that $f_n \not\to 0$ uniformly on $[0, 1]$, with $\epsilon = 1/2$.

(c) The derivative of $f_n(x)$ is $f_n'(x) = \dfrac{n - n^3x^2}{(1 + n^2x^2)^2}$ which is clearly < 0 for all $x \geq 1$. Thus $f_n(x)$ is nonincreasing for all $x \geq 1$, so in particular $f_n(1) \geq f_n(x) \geq 0$ for all $x \geq 1$. But $f_n(1) = n/(1+n^2) \to 0$, hence $f_n(x) \to 0$ uniformly for all $x \geq 1$.

24.17. Assume that $f_n \to f$ uniformly and f_n is continuous. By Theorem 24.3 it follows that f is continuous. Assuming $x_n \to x$, we are asked to prove that $\lim f_n(x_n) = f(x)$. In other words, given $\epsilon > 0$, we must show that $|f_n(x_n) - f(x)| < \epsilon$ for all sufficiently large n. By the Triangle Inequality we have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \tag{1}$$

for all n. Since $f_n \to f$ uniformly, we can find N_1 such that $|f_n(z)-f(z)| < \epsilon/2$ holds for all $n > N_1$ and all z. In particular $|f_n(x_n) - f(x_n)| < \epsilon/2$ whenever $n > N_1$. On the other hand, since f is continuous and $x_n \to x$, we can find N_2 such that $|f(x_n) - f(x)| < \epsilon/2$ whenever $n > N_2$. Letting $N = \max(N_1, N_2)$, we see from (1) that $|f_n(x_n) - f(x)| < \epsilon$ whenever $n > N$. This completes the proof.

25.3. Let $f_n(x) = \dfrac{n + \cos x}{2n + \sin^2 x}$.

(a) Clearly $\lim f_n(x) = 1/2$ for all x, so the limit function is $f(x) = 1/2$. Since $|\cos x| \leq 1$ and $0 \leq \sin^2 x \leq 1$, we have

$$\left| f_n(x) - \frac{1}{2} \right| = \left| \frac{n + \cos x}{2n + \sin^2 x} - \frac{1}{2} \right| = \left| \frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)} \right| \leq \frac{3}{4n} \to 0$$

and this is independent of x. Thus $f_n \to 1/2$ uniformly.

(b) By part (a) and Theorem 25.2 we have

$$\lim_{n\to\infty} \int_2^7 f_n(x)\,dx = \int_2^7 \lim_{n\to\infty} f_n(x)\,dx = \int_2^7 \frac{1}{2}\,dx = \frac{5}{2}.$$

25.6. (a) Assume that $\sum_{k=0}^{\infty} |a_k| < \infty$. Then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for all $x \in [-1, 1]$, by the Weierstrass M-test. (This is because $|a_k x^k| \leq |a_k|$ for all $x \in [-1, 1]$.) Moreover, the partial sums $\sum_{k=0}^{n} a_k x^k$ are polynomials, hence they are continuous on $[-1, 1]$, so by Theorem 24.3 the limit function $\sum_{k=0}^{\infty} a_k x^k$ is continuous on $[-1, 1]$.

(b) For example, we know that $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ (this is the p-series with $p = 2$). Therefore, it follows by part (a) that the power series $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ is continuous on $[-1, 1]$.

26.2. (a) We have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for all $x \in (-1, 1)$ (geometric series). Differentiating term by term according to Theorem 26.5, we see that

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

for all $x \in (-1,1)$. Multiplying through by x, we see that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

for all $x \in (-1,1)$.

(b) Plugging $x = 1/2$ into the result of part (a), we see that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n(\tfrac{1}{2})^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

(c) Plugging $x = 1/3$ and $x = -1/3$ into the result of part (a), we see that

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n(\tfrac{1}{3})^n = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \sum_{n=1}^{\infty} n(-\tfrac{1}{3})^n = \frac{-\frac{1}{3}}{(1-(-\frac{1}{3}))^2} = -\frac{3}{16}.$$

26.6. Let

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and note that the interval of convergence for these series is $(-\infty, \infty)$.

(a) Differentiating term by term according to Theorem 26.5, we get

$$s'(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots = c(x)$$

and

$$c'(x) = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -s(x).$$

(b) By part (a) plus the usual laws of differentiation, we have $(s^2+c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0$. From this it follows that $s^2 + c^2$ is a constant function.

(c) By part (b) $s^2 + c^2$ is a constant function, say $s^2 + c^2 = K$ for some constant K. Plugging in $x = 0$ we get $s(0)^2 + c(0)^2 = 1^2 + 0^2 = 1$, hence $K = 1$. This proves that $s(x)^2 + c(x)^2 = 1$ for all x.

26.7. Let $f(x) = |x|$. This function is not differentiable at $x = 0$, because the left-hand derivative is -1 and the right-hand derivative is 1. Therefore, this function is not representable as a power series $\sum_{n=0}^{\infty} a_n x^n$. If it were representable in this way, it would be be differentiable, by Theorem 26.5.