Math 312, Intro. to Real Analysis: Homework #7 Solutions

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The assignment consists of Exercises 20.1, 20.18, 23.1, 23.4, 23.6, 24.1, 24.2, 24.6, 24.14, 24.17, 25.3, 25.6, 26.2, 26.6, 26.7 in the Ross textbook. Each exercise counts 5 points.

20.1. For $f(x) = x/|x| = \operatorname{sgn} x$ we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(x) = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f(x) = -1.$$

20.18. For $f(x) = \dfrac{\sqrt{1+3x^2}-1}{x^2}$ and $x \neq 0$ we have

$$f(x) = \frac{(1+3x^2)-1}{x^2(\sqrt{1+3x^2}+1)} = \frac{3}{\sqrt{1+3x^2}+1}$$

hence

$$\lim_{x\to 0} f(x) = \frac{3}{2}.$$

23.1. (a) $R = 1$, $I = (-1, 1)$.

(b) $R = \infty$, $I = (-\infty, \infty)$.

(c) $R = 1/2$, $I = [-1/2, 1/2]$.

(d) $R = 3$, $I = (-3, 3)$.

(e) $R = \infty$, $I = (-\infty, \infty)$.

(f) $R = 2$, $I = [-2, 2]$.

(g) $R = 4/3$, $I = [-4/3, 4/3)$.

(h) $R = 4$, $I = [-4, 4]$.

23.4. Note that $a_n = (6/5)^n$ if n is even, $(2/5)^n$ if n is odd.

Moreover $a_{n+1}/a_n = 2/(5 \cdot 3^n)$ if n is even, $(2 \cdot 3^n)/5$ if n is odd.

(a) $\limsup a_n^{1/n} = 6/5$, $\liminf a_n^{1/n} = 2/5$,

$\limsup a_{n+1}/a_n = \infty$, $\liminf a_{n+1}/a_n = 0$.

(b) $\sum a_n$ and $\sum(-1)^n a_n$ diverge, because $a_n \not\to 0$.

(c) For the power series $\sum a_n x^n$ we have $R = 5/6$ and $I = (-5/6, 5/6)$.

23.6. Suppose $a_n \geq 0$ and $x \geq 0$. If $\sum a_n x^n$ converges, then $\sum a_n(-x)^n$ converges absolutely (because $|a_n(-x)^n| = a_n x^n$), hence $\sum a_n(-x)^n$ converges in view of Corollary 14.7.

24.1. For $f_n(x) = \dfrac{1 + 2\cos^2 nx}{\sqrt{n}}$ we have $|\cos^2 nx| \leq 1$, hence $|f_n(x)| \leq \dfrac{3}{\sqrt{n}}$, hence $|f_n(x)| < \epsilon$ whenever $n > 9/\epsilon^2$. Since $9/\epsilon^2$ is independent of x, we see that f_n converges uniformly to 0.

24.2. For $f_n(x) = x/n$ we have:

(a) $f(x) = \lim f_n(x) = \lim \dfrac{x}{n} = 0$ (pointwise convergence).

(b) $f_n \to 0$ uniformly on $[0,1]$. This is because for all $x \in [0,1]$ we have

$$|f_n(x)| = \frac{|x|}{n} \leq \frac{1}{n} < \epsilon$$

whenever $n > 1/\epsilon$.

(c) $f_n \not\to 0$ uniformly on $[0,\infty)$. This is because for any n we can find $x \in [0,\infty)$ such that $|f_n(x)| \geq 1$. (An example of such an x is $x = n$.) Thus the definition of uniform convergence fails for $\epsilon = 1$.

24.6. Let $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ and $f(x) = x^2$.

(a) Clearly for all x we have $f_n(x) \to f(x)$, i.e., pointwise convergence.

(b) For $x \in [0,1]$ we have

$$|f_n(x) - f(x)| = \left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|-\frac{2x}{n} + \frac{1}{n^2}\right| \leq \frac{2}{n} + \frac{1}{n^2} \to 0$$

independently of $x \in [0,1]$. Thus $f_n \to f$ uniformly on $[0,1]$.

24.14. Let $f_n(x) = \dfrac{nx}{1 + n^2x^2}$.

(a) For $x = 0$ we have $f_n(x) = 0$ for all n, hence $\lim f_n(0) = 0$. For $x \neq 0$ we have

$$f_n(x) = \frac{1}{\frac{1}{nx} + nx} \to 0$$

so again $\lim f_n(x) = 0$. Thus $f_n \to 0$ pointwise for all x.

(b) For all n we have $f_n(1/n) = 1/2$. Therefore, since $1/n \in [0,1]$, we see that $f_n \not\to 0$ uniformly on $[0,1]$, with $\epsilon = 1/2$.

(c) The derivative of $f_n(x)$ is $f_n'(x) = \dfrac{n - n^3x^2}{(1 + n^2x^2)^2}$ which is clearly < 0 for all $x \geq 1$. Thus $f_n(x)$ is nonincreasing for all $x \geq 1$, so in particular $f_n(1) \geq f_n(x) \geq 0$ for all $x \geq 1$. But $f_n(1) = n/(1+n^2) \to 0$, hence $f_n(x) \to 0$ uniformly for all $x \geq 1$.

24.17. Assume that $f_n \to f$ uniformly and f_n is continuous. By Theorem 24.3 it follows that f is continuous. Assuming $x_n \to x$, we are asked to prove that $\lim f_n(x_n) = f(x)$. In other words, given $\epsilon > 0$, we must show that $|f_n(x_n) - f(x)| < \epsilon$ for all sufficiently large n. By the Triangle Inequality we have

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \tag{1}$$

for all n. Since $f_n \to f$ uniformly, we can find N_1 such that $|f_n(z) - f(z)| < \epsilon/2$ holds for all $n > N_1$ and all z. In particular $|f_n(x_n) - f(x_n)| < \epsilon/2$ whenever $n > N_1$. On the other hand, since f is continuous and $x_n \to x$, we can find N_2 such that $|f(x_n) - f(x)| < \epsilon/2$ whenever $n > N_2$. Letting $N = \max(N_1, N_2)$, we see from (1) that $|f_n(x_n) - f(x)| < \epsilon$ whenever $n > N$. This completes the proof.

25.3. Let $f_n(x) = \dfrac{n + \cos x}{2n + \sin^2 x}$.

(a) Clearly $\lim f_n(x) = 1/2$ for all x, so the limit function is $f(x) = 1/2$. Since $|\cos x| \le 1$ and $0 \le \sin^2 x \le 1$, we have

$$\left|f_n(x) - \frac{1}{2}\right| = \left|\frac{n + \cos x}{2n + \sin^2 x} - \frac{1}{2}\right| = \left|\frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)}\right| \le \frac{3}{4n} \to 0$$

and this is independent of x. Thus $f_n \to 1/2$ uniformly.

(b) By part (a) and Theorem 25.2 we have

$$\lim_{n\to\infty} \int_2^7 f_n(x)\,dx = \int_2^7 \lim_{n\to\infty} f_n(x)\,dx = \int_2^7 \frac{1}{2}\,dx = \frac{5}{2}.$$

25.6. (a) Assume that $\sum_{k=0}^{\infty} |a_k| < \infty$. Then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for all $x \in [-1,1]$, by the Weierstrass M-test. (This is because $|a_k x^k| \le |a_k|$ for all $x \in [-1,1]$.) Moreover, the partial sums $\sum_{k=0}^{n} a_k x^k$ are polynomials, hence they are continuous on $[-1,1]$, so by Theorem 24.3 the limit function $\sum_{k=0}^{\infty} a_k x^k$ is continuous on $[-1,1]$.

(b) For example, we know that $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ (this is the p-series with $p = 2$). Therefore, it follows by part (a) that the power series $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ is continuous on $[-1,1]$.

26.2. (a) We have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for all $x \in (-1,1)$ (geometric series). Differentiating term by term according to Theorem 26.5, we see that

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

for all $x \in (-1,1)$. Multiplying through by x, we see that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

for all $x \in (-1,1)$.

(b) Plugging $x = 1/2$ into the result of part (a), we see that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n(\tfrac{1}{2})^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

(c) Plugging $x = 1/3$ and $x = -1/3$ into the result of part (a), we see that

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n(\tfrac{1}{3})^n = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \sum_{n=1}^{\infty} n(-\tfrac{1}{3})^n = \frac{-\frac{1}{3}}{(1-(-\frac{1}{3}))^2} = -\frac{3}{16}.$$

26.6. Let

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and note that the interval of convergence for these series is $(-\infty, \infty)$.

(a) Differentiating term by term according to Theorem 26.5, we get

$$s'(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots = c(x)$$

and

$$c'(x) = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -s(x).$$

(b) By part (a) plus the usual laws of differentiation, we have $(s^2+c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0$. From this it follows that $s^2 + c^2$ is a constant function.

(c) By part (b) $s^2 + c^2$ is a constant function, say $s^2 + c^2 = K$ for some constant K. Plugging in $x = 0$ we get $s(0)^2 + c(0)^2 = 1^2 + 0^2 = 1$, hence $K = 1$. This proves that $s(x)^2 + c(x)^2 = 1$ for all x.

26.7. Let $f(x) = |x|$. This function is not differentiable at $x = 0$, because the left-hand derivative is -1 and the right-hand derivative is 1. Therefore, this function is not representable as a power series $\sum_{n=0}^{\infty} a_n x^n$. If it were representable in this way, it would be be differentiable, by Theorem 26.5.