Math 312, Intro. to Real Analysis: Homework #6 Solutions

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The assignment consists of Exercises 17.3(a,b,c,f), 17.4, 17.9(c,d), 17.10(a,b), 17.14, 18.5, 18.7, 19.1, 19.2(b,c), 19.5 in the Ross textbook. Each exercise counts 10 points.

17.3. (a) By Theorem 17.4(ii) applied three times, $\cos^4 x$ is continuous. Then, by Theorem 17.4(i), $1 + \cos^4 x$ is continuous. Then, by Theorem 17.5, $\log_e(1+\cos^4 x)$ is continuous. Since $1+\cos^4 x \geq 1$ for all x, the domain of this function is the entire real line.

(b) Since $x^\pi = e^{\pi \log x}$, it is clear by Theorem 17.4 that x^π is continuous. It then follows as usual that $(\sin^2 x + \cos^6 x)^\pi$ is continuous. Again, the domain of this function is the entire real line.

(c) We have $2^{x^2} = e^{x^2 \log 2}$ so as before this is continuous. Again, the domain is the entire real line.

(f) By Theorem 17.4(iii), $1/x$ is continuous for all $x \neq 0$. It then follows as usual that $x \sin(1/x)$ is continuous for all $x \neq 0$.

17.4. Example 5 in §8 says that if $\lim x_n = x$ and $x_n \geq 0$ for all n, then $\lim \sqrt{x_n} = \sqrt{x}$. According to Definition 17.1 this says precisely that $\sqrt{x}$ is continuous for all $x \geq 0$.

17.9. (c) Let $f(x) = x\sin(1/x)$ for $x \neq 0$, and let $f(0) = 0$. We wish to show that f is continuous at 0 using the ϵ-δ definition of continuity. Given $\epsilon > 0$, we must find $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $|x| < \delta$. We have $|f(x)| = |x| \cdot |\sin(1/x)| \leq |x|$ since $|\sin y| \leq 1$ for all y. So let $\delta = \epsilon$. Then clearly $|x| < \delta$ implies $|x| < \epsilon$ which implies $|f(x)| < \epsilon$, Q.E.D.

(d) Let $g(x) = x^3$. We wish to show that g is continuous at x_0, for an arbitrary x_0. Given $\epsilon > 0$ and x_0, we must find $\delta > 0$ (depending on ϵ and x_0) such that $|g(x) - g(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. For this particular function $g(x)$ we have $|g(x) - g(x_0)| = |x^3 - x_0^3| = |x^2 + xx_0 + x_0^2| \cdot |x - x_0|$. Even though we have not yet chosen our δ, we may safely assume that $\delta \leq 1$. Then $|x - x_0| < \delta$ implies $|x| < |x_0| + 1$ which implies

$$|x^2+xx_0+x_0^2| \le (|x_0|+1)^2+(|x_0|+1)|x_0|+|x_0|^2 = 3|x_0|^2+3|x_0|+1$$

so let $M = 3|x_0|^2+3|x_0|+1$. Then $|g(x)-g(x_0)| \le M|x-x_0|$, so it will work to choose $\delta = \min(1, \epsilon/M)$. Note that M depends on x_0, so δ depends on both ϵ and x_0.

17.10. (a) We have $f(x) = 1$ for all $x > 0$, and $f(0) = 0$. Thus $\lim 1/n = 0$ but $\lim f(1/n) = 1 \neq 0 = f(0)$, so f is disconinuous at 0 in view of Definition 17.1.

(b) We have $g(x) = \sin(1/x)$ for all $x \neq 0$, and $g(0) = 0$. Letting $a_n = (n+\frac{1}{2})\pi$ we see that $\lim 1/a_n = 0$ but $g(1/a_n) = \sin a_n = (-1)^n$ so $\lim g(1/a_n)$ does not exist. Thus g is discontinuous at 0 in view of Definition 17.1.

17.14. Define $f(x) = 0$ if x is irrational, and $f(r) = 1/n$ for all rational numbers $r = m/n$ where $n > 0$ and $\mathrm{GCD}(m,n) = 1$. We want to show that f is continuous at x if x is irrational, and discontinuous at r if r is irrational. We opt to use the ϵ-δ definition of continuity.

Consider what happens near x when x is irrational. Given $\epsilon > 0$, consider the set G_ϵ consisting of all numbers r such that $f(r) \ge \epsilon$. For all $r \in G_\epsilon$ we have $r = m/n$ for some $n \le 1/\epsilon$. Thus G_ϵ is a discrete set of rational numbers. (Here *discrete* means that G_ϵ contains only a finite number of points in each interval.) Consequently, since $x \notin G_\epsilon$, we can find an open interval (a,b) which contains x and is disjoint from G_ϵ. On this interval (a,b) we have $f(z) \le \epsilon$ for all z in the interval. Thus, letting $\delta = \min(|x-a|, |x-b|)$, we have $\delta > 0$ and $|f(z)| < \epsilon$ whenever $|z-x| < \delta$. Since $f(x) = 0$, we see that f is continuous at x.

On the other hand, consider what happens near r when r is rational. We have $f(r) = 1/n > 0$. Let $\epsilon = f(r)$ and consider any $\delta > 0$. Clearly there are irrational numbers x in the open interval $r-\delta < x < r+\delta$ and for these x's we have $f(x) = 0$. Thus $|f(x)-f(r)| = \epsilon$ even though $|x-r| < \delta$. This shows that f is discontinous at r.

18.5. (a) Let f and g be continuous on $[a,b]$ and assume $f(a) \ge g(a)$ and $f(b) \le g(b)$. Consider the function $h(x) = f(x) - g(x)$. Clearly h is continuous on $[a,b]$ and $h(a) \ge 0$ and $h(b) \le 0$. By the Intermediate Value Theorem, there is at least one $x \in [a,b]$ such that $h(x) = 0$. Then clearly $f(x) = g(x)$.

(b) Example 1 on page 128 may be viewed as the special case of part (a) where $[a,b] = [0,1]$ and $f(x) \in [0,1]$ and $g(x) = x$ for all $x \in [0,1]$.

18.7. Let $f(x) = x2^x$. Clearly $f(x)$ is continuous for all x, and $f(0) = 0$ and $f(1) = 2$. Since $0 < 1 < 2$, it follows by the Intermediate Value Theorem that $f(x) = 1$ for some $x \in [0,1]$. Clearly $x \neq 0, 1$, so $x \in (0,1)$.

19.1. (a) Uniformly continuous, by Theorem 19.2.

(b) Uniformly continuous, by Theorem 19.2.

(c) Uniformly continuous, by part (b).

(d) x^3 is not uniformly continous, because $|x^3 - (x+\delta)^3| > 3x^2\delta > 3/\delta$ whenever $x > 1/\delta$, no matter how small δ is.

(e) $1/x^3$ is not uniformly continouous on $(0, 1]$, by Theorem 19.5.

(f) $\sin(1/x^2)$ is not uniformly continous on $(0, 1]$, by Theorem 19.5.

(g) $f(x) = x^2 \sin(1/x)$ is uniformly continouus on $(0, 1]$ by Theorem 19.5, because it can be extended to a continous function on $[0, 1]$ by defining $f(0) = 0$.

19.2. (b) For $x, y \in [0, 3]$ we have $|x^2 - y^2| = |x + y||x - y| \leq 6|x - y|$. Consequently, given $\epsilon > 0$, we have $|x^2 - y^2| < \epsilon$ whenever $|x - y| < \epsilon/6$ and $x, y \in [0, 3]$. Letting $\delta = \epsilon/6$ we see directly that x^2 is uniformly continuous on $[0, 3]$, in terms of the ϵ-δ definition of uniform continuity.

(c) For $x, y \geq 1/2$ we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{xy} \leq 4|x - y|.$$

Consequently, given $\epsilon > 0$, we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon$$

whenever $|x - y| < \epsilon/4$ and $x, y \geq 1/2$. Letting $\delta = \epsilon/4$ we see directly that $1/x$ is uniformly continuous on $[1/2, \infty)$, in terms of the ϵ-δ definition of uniform continuity.

19.5. (a) Uniformly continuous, by Theorem 19.5.

(b) Not uniformly continuous, by Exercise 19.4(a).

(c) Not uniformly continuous, by Exercise 19.4(a).

(d) Not uniformly continuous, by Exercise 19.4(a).

(e) Not uniformly continuous, by Exercise 19.4(a).

(f) Uniformly continuous, by Theorem 19.6.