Math 312, Intro. to Real Analysis: Midterm Exam #2 Solutions

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1. True or False (2 points each)

   (a) Every monotone sequence of real numbers is convergent.
   False.

   (b) Every sequence of real numbers has a $\limsup$ and a $\liminf$.
   True.

   (c) Every sequence of real numbers has a monotone subsequence.
   True.

   (d) Every sequence of real numbers has a convergent subsequence.
   False.

   (e) If $\liminf a_n = \limsup a_n = \alpha$ then $\lim a_n = \alpha$.
   True.

   (f) We can find a sequence of real numbers, (a_n), such that the subsequential limits of (a_n) are exactly the real numbers in the closed interval $[-1, 1]$.
   True. An example is the sequence

   $$-1, 0, 1, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots.$$

   (g) The series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ is absolutely convergent.
   True.

   (h) If $\sum |a_n|$ is convergent, then so is $\sum |a_n|^2$.
   True. Here is a proof. If $\sum |a_n| < \infty$, then $\lim |a_n| = 0$, hence in particular we can find an N such that $|a_n| < 1$ for all $n > N$. Then, for all $n > N$ we have $|a_n|^2 < |a_n|$, so $\sum |a_n|^2$ converges by comparison with $\sum |a_n|$.

   (i) If $\sum a_n$ is convergent, then so is $\sum a_n^2$.
   False. An example is the series

   $$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

   which is convergent by the alternating series test, but the series

   $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

   is divergent.

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p \geq 1$.
   False. However, it is true for $p > 1$.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{9}$.

True. This is a geometric series.

(l) The function $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

True.

(m) If a function is uniformly continuous on the interval $(a, b]$ and on the interval $[b, c)$, then it is uniformly continuous on the interval (a, c).

True.

2. (6 points each) Which of the following series are convergent and/or absolutely convergent? Please indicate which tests you are using and show your work.

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Solution. Convergent by the alternating series test. Not absolutely convergent, because it is known that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent (it is the p-series with $p = 1$).

(b) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n^2 + n} - n)$

Solution. $\lim(\sqrt{n^2 + n} - n) = 1/2 \neq 0$, so the series is divergent by the nth term test.

(c) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n^2 + 1} - n)$

Solution. We have $\sqrt{n^2 + 1} - n = \dfrac{1}{\sqrt{n^2 + 1} + n} > \dfrac{1}{3n}$, so the given series diverges by comparison with the divergent series $\sum \dfrac{1}{3n}$.

(d) $\displaystyle\sum \frac{1}{1.01^n - 1000}$

Solution. Converges absolutely, by the ratio test.

(e) $\displaystyle\sum \frac{1}{n^2 + \sqrt{n^2 + 1}}$

Solution. Converges absolutely, by comparison with the convergent series $\sum \dfrac{1}{n^2}$ (the p-series with $p = 2$).

(f) $\displaystyle\sum_{n=1}^{\infty} \log \frac{n+1}{n}$

Solution. Note that $\log \dfrac{n+1}{n} = \log(n+1) - \log n$. By cancelling terms, the nth partial sum of our series is

$$(\log 2 - \log 1) + (\log 3 - \log 2) + \cdots + (\log(n+1) - \log n) \quad = \quad \log(n+1).$$

Since the limit of the nth partial sum is $+\infty$, our series is divergent.

3. (8 points) Use algebra plus limit laws to calculate

$$\lim \frac{\log \sqrt{e^{(2n+11)/n}}}{\sin((4n\pi + 5)/16n)}.$$

In performing this calculation, you may take it for granted that functions such as $\sin x$, $\cos x$, e^x, $\log x$, $\sqrt{x}$, etc. are continuous. Please show your work.

Solution. In the numerator we have $\lim(2n+11)/n = 2$, hence $\lim e^{(2n+11)/2} = e^2$, hence $\lim \log\sqrt{e^{(2n+11)/2}} = \log\sqrt{e^2} = \log e = 1$. In the denominator we have $\lim(4n\pi+5)/16n = \pi/4$, hence $\lim \sin((4n\pi+5)/16n) = \sin(\pi/4) = 1/\sqrt{2}$. It follows that the limit of the given fraction is $\sqrt{2}$.

4. (5 points) Assume that $f(x)$ is continuous on the closed interval $[0,1]$. Assume also that $f(x) \in [0,1]$ for all $x \in [0,1]$. Using known theorems about continuous functions, prove that the equation $f(x) = x$ has at least one solution in $[0,1]$.

   Solution. If $f(0) = 0$ or $f(1) = 1$, there is nothing to prove. Otherwise, we have $f(0) > 0$ and $f(1) < 1$. Therefore, letting $g(x) = f(x) - x$, we have $g(0) > 0$ and $g(1) < 0$. By the Intermediate Value Theorem applied to the continuous function g, there is some c such that $0 < c < 1$ and $g(c) = 0$. Hence $f(c) = c$, Q.E.D.

5. (3 points each) Which of the following functions are continous and/or uniformly continuous on the specified domain?

   (a) $1/x$ on its natural domain.

   Solution. The natural domain of our function is $(-\infty, 0) \cup (0, +\infty)$. Our function is continuous but not uniformly continuous on this domain.

   (b) $1/x$ on $(1, \infty)$.

   Solution. Uniformly continuous (hence continous).

   (c) $x \cos \dfrac{1}{x}$ on $(0, 10]$.

   Solution. Uniformly continuous (hence continuous).

   (d) $\sqrt{x}$ on $[0, \infty)$.

   Solution. Uniformly continuous (hence continuous).

   (e) $f(x) = \begin{cases} |x|/x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$ on $(-\infty, \infty)$.

   Solution. Not continuous, hence not uniformly continuous.

6. (10 points) It is known that the function $\sqrt{x}$ is uniformly continuous on the interval $[0.01, 100]$. Given $\epsilon > 0$, find a $\delta > 0$ (depending only on ϵ) such that $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $0.01 \le x < y \le 100$ and $|x - y| < \delta$.

   Solution. We want $|\sqrt{x} - \sqrt{y}| < \epsilon$. Note that

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \le \frac{|x-y|}{0.1 + 0.1} = 5|x - y|$$

   throughout the interval $[0.01, 100]$, since $x \ge 0.01$ and $y \ge 0.01$ on that interval. Thus, letting $\delta = \epsilon/5$, we see that $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $|x - y| < \delta$.