Math 312, Intro. to Real Analysis: Homework #5 Solutions

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The assignment consists of Exercises 14.3, 14.4, 14.6, 14.13, 15.3, 15.4, 15.7 in the Ross textbook. Each problem counts 10 points.

In solving some of these exercises, we use the fact that $\sum 1/n^s$ is convergent if $s > 1$ and divergent if $s \leq 1$. This result appears as Theorem 15.1 in the textbook.

14.3. (a) $\sum \frac{1}{\sqrt{n!}}$ converges by the ratio test: $\frac{a_{n+1}}{a_n} = \frac{1}{\sqrt{n+1}} \to 0$.

(b) $\sum \frac{2+\cos n}{3^n}$ converges by comparision with the geometric series $\sum \frac{3}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \cdots = \frac{3}{2}$, because $1 \leq 2 + \cos x \leq 3$.

(c) $\sum 1/(2^n + n)$ converges by comparison with the geometric series $\sum 1/2^n$, because $2^n + n \geq 2^n$ for all n.

(d) $\sum \left(\frac{1}{2}\right)^n (50 + \frac{2}{n})$ converges by comparison with the geometric series $\sum 52 \left(\frac{1}{2}\right)^n$, because $50 + \frac{2}{n} \leq 52$.

(e) $\sum \sin(n\pi/9)$ diverges by Corollary 14.5, because $\lim \sin(n\pi/9)$ does not exist.

(f) $\sum \frac{100^n}{n!}$ converges by the ratio test: $\frac{a_{n+1}}{a_n} = \frac{100}{n+1} \to 0$.

14.4. (a) $\sum 1/(n+(-1)^n)^2$ converges by comparison with the convergent series $\sum 1/(n-1)^2$, because $(n+(-1)^n)^2 \geq (n-1)^2$ for all n.

(b) $\sum \left(\sqrt{n+1} - \sqrt{n}\right) = \sum 1/\left(\sqrt{n+1} + \sqrt{n}\right)$ diverges by comparison with the divergent series $\sum 1/2\sqrt{n+1}$, because $\sqrt{n+1} + \sqrt{n} \leq 2\sqrt{n+1}$ for all n.

(c) $\sum \frac{n!}{n^n}$ converges by the ratio test: $\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1$.

14.6. Assume that $\sum |a_n|$ converges and that (b_n) is a bounded sequence. Let M be such that $|b_n| \leq M$ for all n. Then $|a_n b_n| = |a_n||b_n| \leq M|a_n|$.

Hence $\sum |a_n b_n|$ is convergent, by comparison with the convergent series $\sum M|a_n| = M\sum |a_n|$. Thus $\sum a_n b_n$ is absolutely convergent. It follows by Corollary 14.7 that $\sum a_n b_n$ is convergent.

14.13. (a) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} ar^n$ where $a = \frac{2}{3}$ and $r = \frac{2}{3}$. By equation (2) on page 91, the sum of this geometric series is $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \dfrac{\frac{2}{3}}{1-\frac{2}{3}} = 2$.

Similarly with $a = r = -\frac{2}{3}$ we have $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \dfrac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{2}{5}$.

(b) By cancellation, the nth partial sum is

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1}-\frac{1}{n}\right) + \left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

for all n. Taking the limit as $n \to \infty$ we obtain $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

(c) By cancellation, the nth partial sum is

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k-1}{2^{k+1}} &= \sum_{k=1}^{n} \left(\frac{k}{2^k} - \frac{k+1}{2^{k+1}}\right) \\
&= \left(\frac{1}{2}-\frac{2}{4}\right) + \left(\frac{2}{4}-\frac{3}{8}\right) + \cdots + \left(\frac{n-1}{2^{n-1}}-\frac{n}{2^n}\right) + \left(\frac{n}{2^n}-\frac{n+1}{2^{n+1}}\right) \\
&= \frac{1}{2} - \frac{n+1}{2^{n+1}}
\end{aligned}$$

for all n. Taking the limit as $n \to \infty$ we obtain $\sum_{k=1}^{\infty} \frac{k-1}{2^{k+1}} = \frac{1}{2}$.

(d) Using the result of part (c) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{2^n} &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots \\
&= 4 \cdot \left(\frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \cdots\right) \\
&= 4 \cdot \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} \\
&= 4 \cdot \frac{1}{2} \;=\; 2.
\end{aligned}$$

15.3. Letting $u = \log x$ we have $du = dx/x$. For $s \neq 1$ we have

$$\int_2^{\infty} \frac{dx}{x(\log x)^s} = \int_{\log 2}^{\infty} \frac{du}{u^s} = \left.\frac{u^{-s+1}}{-s+1}\right|_{\log 2}^{\infty}$$

and this is finite if $s > 1$, infinite if $s < 1$. Thus by the integral test we see that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$ converges if $s > 1$ and diverges if $s < 1$. For the special case $s = 1$ we have

$$\int_2^{\infty} \frac{dx}{x\log x} = \int_{\log 2}^{\infty} \frac{du}{u} = \left.\log u\right|_{\log 2}^{\infty}$$

and this is infinite, hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges by the integral test.

15.4. (a) $\sum 1/\sqrt{n} \log n$ diverges by comparison with the divergent series $\sum 1/n$.

This is because $\log n < \sqrt{n}$, hence $\sqrt{n} \log n < n$.

(b) $\sum \log n/n$ diverges by comparison with the divergent series $\sum 1/n$.

This is because $\log n/n > 1/n$ for all n.

(c) Letting $u = \log \log x$ we have $du = dx/x \log x$, hence

$$\int_4^\infty \frac{dx}{x \log x \log \log x} = \int_{\log \log 4}^\infty \frac{du}{u} = \log u\Big|_{\log \log 4}^{\infty} = \infty$$

hence $\sum 1/n \log n \log \log n$ diverges by the integral test.

(d) $\sum \log n/n^2$ converges by comparison with the convergent series $\sum 1/n^{3/2}$.

This is because $\log n < \sqrt{n}$, hence $\log n/n^2 < 1/n\sqrt{n} = 1/n^{3/2}$.

15.7. Assume that (a_n) is a nonincreasing sequence of real numbers and $\sum a_n$ converges. We are going to prove that $\lim na_n = 0$.

Note first that $a_n \geq 0$ for all n. This is because, if $a_n < 0$ for some n, then since our sequence is nonincreasing we would have $a_m \leq a_n < 0$ for all $m > n$, hence $\lim a_m \neq 0$ which would contradict Corollary 14.5.

Now, given $\epsilon > 0$, apply the Cauchy criterion to find an N so large that $a_m + a_{m+1} + \cdots + a_n < \epsilon$ whenever $N < m < n$. In particular we have

$$a_{N+1} + a_{N+2} + \cdots + a_n < \epsilon$$

and

$$a_{N+1} \geq a_{N+2} \geq \cdots \geq a_n$$

and

$$\frac{n}{2} < n - N$$

whenever $n > 2N$. From these observations it follows that

$$\frac{na_n}{2} < (n-N)a_n < \epsilon$$

whenever $n > 2N$. This shows that $\lim na_n/2 = 0$, and it is then immediate that $\lim na_n = 0$, Q.E.D.