Math 312, Intro. to Real Analysis: Homework #4 Solutions

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The assignment consists of Exercises 10.6, 10.8, 10.10, 11.2, 11.3, 11.6, 11.10, 12.2, 12.12 in the Ross textbook. Each problem counts 10 points.

10.6. (a) Let (s_n) be a sequence of real numbers. Then, whenever m is less than n we have

$$s_m - s_n = (s_m - s_{m+1}) + (s_{m+1} - s_{m+2}) + \cdots + (s_{n-1} - s_n),$$

hence by the Triangle Inequality

$$|s_m - s_n| \le |s_m - s_{m+1}| + |s_{m+1} - s_{m+2}| + \cdots + |s_{n-1} - s_n|.$$

Now, if our sequence is such that $|s_k - s_{k+1}| \le 1/2^k$ for all k, then our inequality becomes

$$|s_m - s_n| \le \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}}$$

and the right-hand side is the beginning of the geometrical series $\sum_{k=m}^{\infty} 1/2^k = 1/2^{m-1}$. Thus we see that $|s_m - s_n| \le 1/2^{m-1}$ for all m and all $n > m$. Thus (s_n) is a Cauchy sequence. Therefore, by one of our theorems, (s_n) is a convergent sequence.

(b) If we assume only that $|s_k - s_{k+1}| < 1/k$ for all k, we cannot conclude that (s_n) is a Cauchy sequence. An example illustrating this is obtained by letting $s_n =$ the nth partial sum of the harmonic series,

$$s_n = H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For this example we have

$$|s_n - s_{n+1}| = \frac{1}{n+1} < \frac{1}{n}$$

but

$$\lim s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad \text{(harmonic series)}$$

so our sequence (s_n) is divergent.

10.8. Assume that (s_n) is a nondecreasing sequence of real numbers. Let σ_n be the average of the first n numbers in our given sequence:

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}.$$

We claim that the sequence (σ_n) is again nondecreasing. To see this, note that $s_1 \le \cdots \le s_n \le s_{n+1}$, hence $s_1 + \cdots + s_n \le n s_{n+1}$. Adding $n(s_1 + \cdots + s_n)$ to both sides, we obtain

$$n(s_1 + \cdots + s_n) + (s_1 + \cdots + s_n) \le n(s_1 + \cdots + s_n) + n s_{n+1},$$

or, in other words,

$$(n+1)(s_1 + \cdots + s_n) \le n(s_1 + \cdots + s_n + s_{n+1}).$$

Dividing both sides by $n(n+1)$, we obtain

$$\frac{s_1 + \cdots + s_n}{n} \le \frac{s_1 + \cdots + s_n + s_{n+1}}{n+1}$$

or, in other words, $\sigma_n \le \sigma_{n+1}$ for all n. This proves our claim.

The conclusion can be restated as follows:

> *If a sequence is nondecreasing, then its sequence of averages is nondecreasing.*

Note: The textbook assumes in addition that s_n is positive for all n. However, this assumption is not needed in order to obtain the conclusion.

10.10. Define a sequence (s_n) inductively by letting $s_1 = 1$ and $s_{n+1} = (s_n + 1)/3$ for all n.

(a) The first few terms of the sequence are $s_1 = 1$ and $s_2 = 2/3$ and $s_3 = 5/9$ and $s_4 = 14/27$.

(b) Obviously $s_1 > 1/2$. If $s_n > 1/2$, it follows that $s_n + 1 > 3/2$, hence $s_{n+1} = (s_n + 1)/3 > (3/2)/3$, hence $s_n > 1/2$. Thus by induction on n we see that $s_n > 1/2$ for all n.

(c) Since $s_n > 1/2$, it follows that

$$s_n - s_{n+1} = s_n - \frac{s_n + 1}{3} = \frac{2}{3} s_n - \frac{1}{3} > \frac{2}{3} \cdot \frac{1}{2} - \frac{1}{3} = 0$$

so $s_n > s_{n+1}$ for all n, i.e., (s_n) is a decreasing sequence.

(d) Since (s_n) is a decreasing sequence which is bounded below (the bound is $1/2$), it follows by one of our theorems that (s_n) is convergent. Let $\alpha = \lim s_n$. Taking the limit of both sides of the equation $s_{n+1} = (s_n + 1)/3$, we obtain $\alpha = (\alpha + 1)/3$. Solving for α we obtain $\alpha = 1/2$. Thus $\lim s_n = 1/2$.

11.2. Let

$$a_n = (-1)^n, \qquad b_n = \frac{1}{n}, \qquad c_n = n^2, \qquad d_n = \frac{6n+4}{7n-3}.$$

Note that the sequence (a_n) looks like $-1, 1, -1, 1, \ldots$.

(a) A monotone subsequence of $(a_n)_n$ is $1, 1, 1, \ldots$. We can describe this subsequence as $(a_{n_k})_k$ where $n_k = 2k$. Thus $a_{n_k} = (-1)^{2k} = 1$ for all k. Alternatively, using n instead of k as the index, we can describe our subsequence as (a_{2n}).

The sequences (b_n), (c_n), and (d_n) are already monotone. (Note that any sequence is considered to be a subsequence of itself.) To see that (d_n) is monotone, use algebra to show directly that

$$\frac{6n+4}{7n-3} > \frac{6(n+1)+4}{7(n+1)-3}$$

for all n.

(b) The subsequential limits of our sequences are -1 and 1 for (a_n), 0 for (b_n), $+\infty$ for (c_n), and $6/7$ for (d_n).

(c) $\limsup a_n = 1$, $\liminf a_n = -1$. $\limsup b_n = \liminf b_n = \lim b_n = 0$. $\limsup c_n = \liminf c_n = \lim c_n = +\infty$. $\limsup d_n = \liminf d_n = \lim d_n = 6/7$.

(d) (a_n) is divergent and its limit is undefined. (b_n) and (d_n) are convergent to 0 and $6/7$ respectively. (c_n) is divergent to $+\infty$.

(e) (a_n), (b_n), and (d_n) are bounded. (c_n) is unbounded.

11.3. Let

$$s_n = \cos\frac{n\pi}{3}, \qquad t_n = \frac{3}{4n+1}, \qquad u_n = \left(-\frac{1}{2}\right)^n, \qquad v_n = (-1)^n + \frac{1}{n}.$$

Note that the sequence (s_n) looks like

$$\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, \ldots.$$

(a) A monotone sbsequence of (s_n) is $-1, -1, -1, \ldots$, which can be described as (s_{6n+3}), i.e., $(\cos(2n+1)\pi)$. The sequence (t_n) is already monotone. A monotone subsequence of (u_n) is (u_{2n}), i.e., $(1/2^{2n})$, which looks like $1/4, 1/16, 1/64, \ldots$. A monotone subsequence of (v_n) is (v_{2n}), i.e., $(1 + 1/2n)$, which looks like $3/2, 5/4, 7/6, \ldots$.

(b) The subsequential limits are $\pm 1, \pm 1/2$ for (s_n), 0 for (t_n), 0 for (u_n), ± 1 for (v_n).

(c) (s_n) and (v_n) are divergent and their limits are undefined. (t_n) and (u_n) are convergent to 0.

(d) (s_n), (t_n), (u_n), and (v_n) are bounded.

11.6. Assume that (c_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n). By equation (3) on page 64, there are functions $f(n)$ and $g(n)$ such that $b_n = a_{f(n)}$ and $c_n = b_{g(n)}$ for all n. It follows that $c_n = a_{f(g(n))}$ for all n. Thus (c_n) is a subsequence of (a_n) via the composite function $f \circ g$.

11.10. We are given the sequence $1, 1/2, 1, 1, 1/2, 1/3, 1/4, 1/3, 1/2, 1, 1, 1/2, \ldots$. The subsequential limits of this sequence are $1, 1/2, 1/3, \ldots, 1/n, \ldots$ and 0. The lim sup is 1 and the lim inf is 0.

12.2. By definition of limits, $\lim s_n = 0$ means that for all $\epsilon > 0$ we can find N so large that $|s_n - 0| < \epsilon$ for all $n > N$. In other words,

for all $\epsilon > 0$ we can find N so large that $|s_n| < \epsilon$ for all $n > N$.

But then, by the definition of lim sup, this condition is precisely equivalent to saying that $\limsup |s_n| \le 0$. Moreover, $|s_n| \ge 0$ automatically, hence $\limsup |s_n| \ge 0$ automatically, for all sequences. Thus, our condition above is equivalent to saying that $\limsup |s_n| = 0$.

We have now proved that $\lim s_n = 0$ if and only if $\limsup |s_n| = 0$.

12.12. Assume that (s_n) is a sequence of real numbers. As in Exercise 10.8, let σ_n be the average of $s_1, \ldots, s_n$.

(a) We shall prove the chain of inequalities

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n. \tag{1}$$

The middle inequality $\liminf \sigma_n \le \limsup \sigma_n$ is obvious and does not require proof. We prove the right-hand inequality $\limsup \sigma_n \le \limsup s_n$. We omit the proof of the left-hand inequality $\liminf s_n \le \liminf \sigma_n$, which is similar.

In order to prove $\limsup \sigma_n \le \limsup s_n$, we follow the hint given in the textbook.

First, given M and N such that $M > N$, we claim that

$$\sup_{n>M} \sigma_n \;\le\; \frac{s_1 + \cdots + s_N}{M} + \sup_{k>N} s_k. \tag{2}$$

To prove (2), it suffices to prove

$$\sigma_n \;\le\; \frac{s_1 + \cdots + s_N}{M} + \sup_{k>N} s_k \tag{3}$$

for each individual $n > M$. Since $n > M > N$, we can break σ_n into two parts:

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n}. \tag{4}$$

Since $n > M$, the first part of the right-hand side of (4) is $<$ the first part of the right-hand side of (3). And obviously, the second part of the right-hand side of (4) is $\leq$ the second part of the right-hand side of (3). Combining these two observations, we see that (3) holds. From this it follows immediately that (2) holds.

Now, fixing N and taking the limit of (2) as M goes to infinity, we see that

$$\limsup \sigma_n \ \leq \ \sup_{k>N} s_k \tag{5}$$

for all N. Finally, taking the limit of (5) as N goes to infinity, we see that

$$\limsup \sigma_n \ \leq \ \limsup s_k$$

and this completes the proof.

(b) In the case when $\lim s_n$ exists, we have $\lim s_n = \liminf s_n = \limsup s_n$, so (1) becomes

$$\liminf s_n = \liminf \sigma_n = \limsup \sigma_n = \limsup s_n = \lim s_n.$$

From this it follows that $\lim \sigma_n$ exists and is equal to $\lim s_n$.

This conclusion can be restated as follows:

If a sequence is convergent, then its sequence of averages is convergent.