Math 312, Intro. to Real Analysis: Homework #3 Solutions

Stephen G. Simpson

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The assignment consists of Exercises 7.3, 7.5, 8.2(b)(e), 8.3, 8.8, 9.3, 9.4, 9.12, 9.18 in the Ross textbook. Each problem counts 10 points.

7.3. (a) 1, (b) 1, (c) 0, (d) 1, (e) does not converge, (f) 1, (g) diverges to $+\infty$, (h) does not converge, (i) 0, (j) 7/2, (k) diverges to $+\infty$, (l) does not converge, (m) 0, (n) does not converge, (o) 0, (p) 2, (q) 0, (r) 1, (s) 4/3, (t) 0.

7.5. (a) $\lim \sqrt{n^2+1} - n = \lim \dfrac{1}{\sqrt{n^2+1}+n} = 0.$

(b) $\lim \sqrt{n^2+n} - n = \lim \dfrac{n}{\sqrt{n^2+n}+n} = \lim \dfrac{1}{\sqrt{1+1/n}+1} = \dfrac{1}{2}.$

(c) $\lim \sqrt{4n^2+n} - 2n = \lim \dfrac{n}{\sqrt{4n^2+n}+2n} = \lim \dfrac{1}{\sqrt{4+1/n}+2} = \dfrac{1}{4}.$

8.2. (b) The limit is 7/3. Given $\epsilon > 0$ we want

$$\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| < \epsilon. \tag{1}$$

This is equivalent to

$$\left| \frac{-106}{9n+21} \right| < \epsilon,$$

i.e.,

$$\frac{106}{9n+21} < \epsilon,$$

i.e.,

$$n > \frac{106}{9\epsilon} - \frac{7}{3}.$$

The above reasoning works in reverse, so we have shown that (1) holds for all $n > N$ where $N = 106/9\epsilon - 7/3$. This proves the limit.

(e) The limit is 0. Given $\epsilon > 0$ we want

$$\left| \frac{\sin n}{n} \right| < \epsilon. \tag{2}$$

Since $|\sin x| \leq 1$ for all x, (2) will hold provided $1/n < \epsilon$, i.e., $n > 1/\epsilon$. Thus we may take $N = 1/\epsilon$ and this proves the limit.

8.3. Assume $\lim s_n = 0$ and $s_n \geq 0$ for all n. Given $\epsilon > 0$, let N be so large that $s_n < \epsilon^2$ for all $n > N$. Then $\sqrt{s_n} < \epsilon$ for all $n > N$. This proves that $\lim \sqrt{s_n} = 0$.

8.8. (a) We want $|\sqrt{n^2+1} - n| < \epsilon$. By algebra this is equivalent to

$$\frac{1}{\sqrt{n^2+1}+n} < \epsilon. \tag{3}$$

But clearly $\sqrt{n^2+1}+n > 2n$, hence (3) will hold provided $1/2n < \epsilon$, i.e., $n > 1/2\epsilon$. This proves the limit.

(b) We want $|\sqrt{n^2+n} - n - 1/2| < \epsilon$. By algebra this is equivalent to

$$\left| \frac{n - \sqrt{n^2+n}}{2\sqrt{n^2+n}+2n} \right| < \epsilon. \tag{4}$$

But clearly $n - \sqrt{n^2+n} < 0$ and $2\sqrt{n^2+n} + 2n > 4n$, so (4) will hold provided

$$\frac{\sqrt{n^2+n}-n}{4n} < \epsilon,$$

i.e.,

$$\frac{\sqrt{1+1/n}-1}{4} < \epsilon,$$

i.e.,

$$n > \frac{1}{(1+4\epsilon)^2-1}.$$

This proves the limit.

(c) We want $|\sqrt{4n^2+n} - n - 1/4| < \epsilon$. By algebra this is equivalent to

$$\left| \frac{2n - \sqrt{4n^2+n}}{4\sqrt{4n^2+n}+8n} \right| < \epsilon. \tag{5}$$

But clearly $2n - \sqrt{4n^2+n} < 0$ and $4\sqrt{4n^2+n} + 8n > 16n$, so (5) will hold provided

$$\frac{\sqrt{4n^2+n}-2n}{16n} < \epsilon,$$

i.e.,

$$\frac{\sqrt{1+1/4n}-1}{8} < \epsilon,$$

i.e.,

$$n > \frac{1}{4(1+8\epsilon)^2 - 4}.$$

This proves the limit.

9.3. Assume $\lim a_n = a$ and $\lim b_n = b$. Then, using limit theorems,

$$\lim \frac{a_n^3 + 4a_n}{b_n^2 + 1} = \frac{\lim(a_n^3 + 4a_n)}{\lim(b_n^2 + 1)} = \frac{\lim a_n^3 + \lim 4a_n}{\lim b_n^2 + 1} = \frac{a^3 + 4a}{b^2 + 1}.$$

9.4. Define a sequence (s_n) inductively by letting $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for all n.

(a) The first four terms are $s_1 = 1$, $s_2 = \sqrt{2}$, $s_3 = \sqrt{\sqrt{2}+1}$,

$$s_4 = \sqrt{\sqrt{\sqrt{2}+1}+1}.$$

(b) Assume that $\lim s_n = \alpha$. By induction on n we can easily show that $s_n \geq 1$ for all n, hence $\alpha \geq 1$. Also, we have $\lim s_{n+1} = \alpha$ because (s_{n+1}) is essentially the same sequence as (s_n). Taking the limit of both sides of the equation $s_{n+1} = \sqrt{s_n + 1}$ and using limit laws, we see that $\alpha = \sqrt{\alpha + 1}$. Hence $\alpha^2 - \alpha - 1 = 0$, i.e., α is a solution of the equation $x^2 - x - 1 = 0$. The solutions of this equation are

$$x = \frac{1 \pm \sqrt{5}}{2}$$

but the solution with $1 - \sqrt{5}$ is ruled out because it is < 0. We conclude that

$$\alpha = \frac{1+\sqrt{5}}{2}.$$

9.12. Assume that $L = \lim |s_{n+1}/s_n|$ exists and that $s_n \neq 0$ for all n.

(a) If $L < 1$, let a be such that $L < a < 1$. Then for all sufficiently large n we have $|s_{n+1}/s_n| < a$. Let N be so large that this holds for all $n \geq N$. Then for all $n > N$ we have

$$s_n = s_N \cdot \frac{s_{N+1}}{s_N} \cdot \frac{s_{N+2}}{s_{N+1}} \cdot \cdots \cdot \frac{s_n}{s_{n-1}},$$

hence

$$|s_n| = |s_N| \cdot \left|\frac{s_{N+1}}{s_N}\right| \cdot \left|\frac{s_{N+2}}{s_{N+1}}\right| \cdot \cdots \cdot \left|\frac{s_n}{s_{n-1}}\right|,$$

hence

$$|s_n| \leq |s_N| \cdot a \cdot a \cdot \dots \cdot a = |s_N| \cdot a^{n-N} = \frac{|s_N|}{a^N} \cdot a^n.$$

Since $\lim a^n = 0$, it follows that $\lim s_n = 0$.

(b) If $L > 1$, let $t_n = 1/|s_n|$. We have $|t_{n+1}/t_n| = 1/|s_{n+1}/s_n|$, hence $\lim |t_{n+1}/t_n| = 1/L < 1$. Applying part (a) to the sequence (t_n) we get $\lim t_n = 0$. It then follows by Theorem 9.10 that $\lim |s_n| = +\infty$.

9.18. (a) For $a \neq 1$ we have

$$\begin{aligned}
&(1-a)(1+a+a^2+\dots+a^n) \\
&= (1+a+a^2+\dots+a^n) - (a+a^2+a^3+\dots+a^{n+1}) \\
&= 1 - a^{n+1}
\end{aligned}$$

hence

$$1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}.$$

(b) For $|a| < 1$ we have $\lim a^{n+1} = 0$ in view of Basic Example 9.7(b). Therefore, from part (a) plus limit laws, we have

$$\lim(1+a+a^2+\dots+a^n) = \frac{1}{1-a}.$$

(c) Letting $a = 1/3$ in part (b), we obtain

$$\lim\left(1 + \frac{1}{3} + \frac{1}{9} + \dots + \frac{1}{3^n}\right) = \frac{1}{1-1/3} = \frac{3}{2}.$$

(d) For $a \geq 1$ we have $1 + a + a^2 + \dots + a^n \geq 1 + n$ for all n, hence

$$\lim(1+a+a^2+\dots+a^n) = +\infty.$$