Math 312, Intro. to Real Analysis: Midterm Exam #1 Solutions

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1. True or False (3 points each)

   (a) Every ordered field has the Archimedean property.
   Answer: False.

   (b) The ordered field axioms imply $|a - b| \leq |a| + |b|$ for all a, b.
   Answer: True.

   (c) If $\lim a_n = -\infty$ then $\limsup a_n = -\infty$.
   Answer: True.

   (d) For any sequence of real numbers, the lim inf and the lim sup always exist and furthermore the lim inf is always $\leq$ the lim sup.
   Answer: True.

   (e) The equation $3x^3 + 2x^2 + 3x + 2 = 0$ has a rational solution.
   Answer: True. A rational solution is $x = -2/3$.

   (f) $\sqrt[3]{216}$ is an irrational number.
   Answer: False. $\sqrt[3]{216} = 6$.

   (g) The limit of a convergent sequence of negative numbers is negative.
   Answer: False. For example, the sequence $-1/n$ for $n = 1, 2, 3, \ldots$ has 0 as its limit.

   (h) The limit of a convergent sequence of rational numbers is rational.
   Answer: False.

   (i) Every interval contains at least three rational numbers.
   Answer: True.

   (j) Every bounded sequence of real numbers is convergent.
   Answer: False.

   (k) Every convergent sequence of real numbers is bounded.
   Answer: True.

   (l) Every monotone sequence of real numbers is convergent.
   Answer: False. For example, the sequence $1, 2, 3, \ldots$ is divergent to ∞.

(m) If (a_n) is a monotone sequence of real numbers, then $\lim a_n$ exists and belongs to the interval $(-\infty, \infty)$.

Answer: False. Same example as above.

2. (7 points each)

(a) Give an example of a sequence of real numbers such that

$$-\infty < \inf a_n < \lim a_n < \sup a_n < \infty.$$

Answer: An example is $a_n = \dfrac{(-1)^n}{n}$. The inf is -1, the sup is $1/2$, and the limit is 0.

(b) Give an example of a sequence of real numbers such that

$$\limsup a_n, \quad \liminf a_n, \quad \sup a_n, \quad \inf a_n$$

are four distinct real numbers.

Answer: An example is $a_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$. The inf is -2, the sup is $1 + 1/2$, the $\liminf$ and $\limsup$ are -1 and 1.

(c) Give an example of a sequence of real numbers such that

$$\liminf a_n = -\infty \quad \text{and} \quad \limsup a_n = \sqrt{2}.$$

Answer: An example is $-1, \sqrt{2}, -2, \sqrt{2}, -3, \sqrt{2}, -4, \ldots$. In other words, $a_{2n} = \sqrt{2}$ and $a_{2n-1} = -n$ for all n.

3. (8 points) It can be shown that $\sqrt[3]{1+\sqrt{5}}$ is an *algebraic number*, i.e., it is a solution of some polynomial equation with integer coefficients. Find such an equation.

Answer: Let $\alpha = \sqrt[3]{1+\sqrt{5}}$. Then $\alpha^3 = 1+\sqrt{5}$, i.e., $\alpha^3 - 1 = \sqrt{5}$ hence $\alpha^6 - 2\alpha^3 + 1 = 5$, i.e., $\alpha^6 - 2\alpha^3 - 4 = 0$. The desired equation is $x^6 - 2x^3 + 4 = 0$.

4. (8 points) Find all candidates for rational solutions of the equation

$$2x^2 - ax + 5 = 0$$

where a is an unspecified integer.

Answer: The candidates are $\pm 5, \pm\dfrac{5}{2}, \pm 1, \pm\dfrac{1}{2}$. This is according to the Rational Zeros Theorem, page 9 in the Ross textbook.

5. (12 points) Use algebra plus limit laws to calculate

$$\lim \frac{\sqrt{2n^2+5n}}{n+4}.$$

Answer:

$$\begin{aligned}
\lim \frac{\sqrt{2n^2+5n}}{n+4} &= \lim \frac{\sqrt{2+\frac{5}{n}}}{1+\frac{4}{n}} = \frac{\lim \sqrt{2+\frac{5}{n}}}{\lim 1+\frac{4}{n}} \\
&= \frac{\sqrt{\lim 2+\frac{5}{n}}}{\lim 1+\frac{4}{n}} = \frac{\sqrt{2+\lim \frac{5}{n}}}{1+\lim \frac{4}{n}} \\
&= \frac{\sqrt{2+0}}{1+0} = \sqrt{2}.
\end{aligned}$$

6. (12 points) It can be shown that

$$\lim \frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} = 1.$$

Given $\epsilon > 0$, find an N so large that

$$\left| \frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} - 1 \right| < \epsilon$$

holds for all $n > N$.

Answer: Given $\epsilon > 0$, we want

$$\left| \frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} \right| < \epsilon$$

and by algebra this is equivalent to saying that

$$\left| \frac{-4001}{\sqrt[3]{n}-1001} \right| < \epsilon.$$

If $n > 1001^3$ then $\sqrt[3]{n} - 1001 > 0$, and $|-4001| = 4001$, so the above is equivalent to saying that

$$\frac{4001}{\sqrt[3]{n}-1001} < \epsilon.$$

By algebra, this is equivalent to

$$\frac{4001}{\epsilon} + 1001 < \sqrt[3]{n}$$

i.e.,

$$n > \left(\frac{4001}{\epsilon} + 1001 \right)^3.$$

Thus we may take

$$N = \left(\frac{4001}{\epsilon} + 1001 \right)^3.$$