Math 312, Intro. to Real Analysis: Homework #2 Solutions

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Wednesday, February 4, 2009

The assignment consists of Exercises 4.1, 4.3, 4.7, 4.14, 5.3 in the Ross textbook. Each problem counts 10 points.

4.1 The sups of these sets are found in Exercise 4.3. In each case, if α is the sup of the set, then α, $\alpha + 1$, and $\alpha + 2$ are three different upper bounds of the set. If the set has no sup, then it has no upper bound.

4.3 The sups are: (a) 1, (b) 1, (c) 7, (d) π, (e) 1, (f) 0, (g) 3, (h) NO SUP, (i) 1, (j) 1, (k) NO SUP, (l) 2, (m) 2, (n) $\sqrt{2}$, (o) 0, (p) 10, (q) 16, (r) 1, (s) 1/2, (t) 2, (u) NO SUP, (v) 1/2, (w) 1/2.

4.7 Let S and T be nonempty bounded subsets of $\mathbb{R}$. The completeness property of $\mathbb{R}$ implies that $\alpha = \sup S$ and $\beta = \sup T$ exist.

(a) Assume $S \subseteq T$. In other words, every x belonging to S also belongs to T.

i. Since β is an upper bound of T, every x belonging to S is $\leq \beta$. In other words, β is an upper bound of S. Since α is the *least* upper bound of S, it follows that $\alpha \leq \beta$. In other words, $\sup S \leq \sup T$.

ii. A similar argument shows that $\inf T \leq \inf S$.

iii. Since S is nonempty, let x be an element of S. By definition of inf and sup we have $\inf S \leq x \leq \sup S$ for all such x.

Combining our results, we have $\inf T \leq \inf S \leq \sup S \leq \sup T$, Q.E.D.

(b) $S \cup T$ is again a nonempty bounded subset of $\mathbb{R}$, so let $\gamma = \sup(S \cup T)$.

i. Since $S \subseteq S \cup T$, it follows[1] that $\alpha \leq \gamma$. Similarly, since $T \subseteq S \cup T$, it follows that $\beta \leq \gamma$. Combining these two inequalities, we have $\max\{\alpha, \beta\} \leq \gamma$.

ii. Conversely, given $\epsilon > 0$, we know that $\gamma - \epsilon < \gamma$, hence $\gamma - \epsilon$ is not an upper bound of $S \cup T$. Therefore, let z be such that $\gamma - \epsilon < z$ and z belongs to $S \cup T$. If z belongs to S, then

[1] by part (a) applied to S and $S \cup T$

$\gamma - \epsilon < z \leq \sup S = \alpha$. Similarly, if z belongs to T, then $\gamma - \epsilon < z \leq \sup T = \beta$. In either case we have $\gamma - \epsilon < \max\{\alpha, \beta\}$. Since this inequality holds for all $\epsilon > 0$, it follows that $\gamma \leq \max\{\alpha, \beta\}$.

Combining these two results, we see that $\gamma = \max\{\alpha, \beta\}$. In other words, $\sup(S \cup T) = \max\{\sup S, \sup T\}$, Q.E.D.

4.14 Let A and B be nonempty bounded subsets of $\mathbb{R}$. Let

$$S = A + B = \{a + b : a \text{ in } A, b \text{ in } B\}.$$

Let $\alpha = \sup A$, $\beta = \sup B$, and $\gamma = \sup(A + B)$.

(a) Let $\epsilon > 0$ be given. Since $\alpha - \epsilon/2 < \alpha = \sup A$, we can find a in A such that $\alpha - \epsilon/2 < a$. Similarly, we can find b in B such that $\beta - \epsilon/2 < b$. Let $c = a + b$. Then $(\alpha + \beta) - \epsilon = (\alpha - \epsilon/2) + (\beta - \epsilon/2) < a + b = c$ and c belongs to $A + B$. It follows that $(\alpha + \beta) - \epsilon < \sup(A + B) = \gamma$. Since this holds for all $\epsilon > 0$, if follows that $\alpha + \beta \leq \gamma$.

(b) Conversely, given c in $A + B$, we can find a in A and b in B such that $c = a + b$. Then $a \leq \sup A = \alpha$ and $b \leq \sup B = \beta$, hence $c = a + b \leq \alpha + \beta$. Thus $\alpha + \beta$ is an upper bound of $A + B$. Since γ is the least upper bound of $A + B$, it follows that $\gamma \leq \alpha + \beta$.

Combining these two results, we see that $\gamma = \alpha + \beta$. In other words, $\sup(A + B) = \sup A + \sup B$.

Similarly, it can be shown that $\inf(A + B) = \inf A + \inf B$.

5.3 For the unbounded sets in 4.1 we have (h) $\inf = 2$, $\sup = \infty$, (k) $\inf = 0$, $\sup = \infty$, (l) $\inf = -\infty$, $\sup = 2$, (o) $\inf = -\infty$, $\sup = 0$, (t) $\inf = -\infty$, $\sup = 2$, (u) $\inf = 0$, $\sup = \infty$. All of the other sets in 4.1 are bounded.