Math 312, Intro. to Real Analysis: Homework #1 Solutions

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The assignment consists of Exercises 1.4, 1.7, 1.11, 2.4, 3.4, 3.7 in the Ross textbook. Each problem counts 10 points.

1.4 (a) $1 = 1$; $1 + 3 = 4$; $1 + 3 + 5 = 9$; $1 + 3 + 5 + 7 = 16$. This pattern suggests that the sum of the first n odd numbers is always n^2. In other words, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all n, and this is our guess.

(b) Let P_n be the proposition $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We shall prove that P_n holds for all n. For $n = 1$ we have P_1 because $1 = 1$. If we assume that P_n holds for a particular n, then for $n + 1$ we have

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2(n + 1) - 1) &= 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) \\ &= n^2 + (2n + 1) \text{ in view of } P_n \\ &= (n + 1)^2 \end{aligned}$$

so this gives us P_{n+1}. By the Induction Principle it follows that P_n holds for all n.

1.7 Let P_n be the proposition that $7^n - 6n - 1$ is divisible by 36. The base of the induction is P_1, which says that $7 - 6 - 1 = 0$ is divisible by 36, which is obvious. Next, assuming P_n for a particular n, we need to prove P_{n+1}, i.e., we need to prove that $7^{n+1} - 6(n + 1) - 1$ is divisible by 36. An algebraic manipulation shows that the expression in question is equal to $36n + 7 \cdot (7^n - 6n - 1)$. Obviously the first term is divisible by 36. Moreover, P_n implies that the second term is divisible by 36. Thus, the whole expression is divisible by 36. We have now shown that P_n implies P_{n+1}. Therefore, by the Induction Principle, P_n holds for all n.

1.11 (a) Assuming that n^2+5n+1 is even, we prove that $(n+1)^2+5(n+1)+1$ is even. An algebraic manipulation shows that the latter expression is equal to $n^2 + 5n + 1 + 2(n + 3)$. Since $n^2 + 5n + 1$ and $2(n + 3)$ are even, it follows that the whole expression is even, Q.E.D.

(b) In actuality, $n^2 + 5n + 1$ is never even.

To see this, consider two cases. If n is even, n^2 and $5n$ are even, hence $n^2 + 5n + 1$ is odd. If n is odd, n^2 and $5n$ are odd, hence $n^2 + 5n$ is even, hence $n^2 + 5n + 1$ is odd. In both cases, $n^2 + 5n + 1$ is odd.

The moral is that, in inductive proofs, we cannot omit the base step. If we omit the base step, we may obtain incorrect conclusions. Part (a) may be viewed as an inductive "proof" that $n^2 + 5n + 1$ is even for all n. This conclusion is incorrect. The "proof" was faulty because the base step $n = 1$ was omitted.

2.4 Let $\alpha = (5 - \sqrt{3})^{1/3}$. Then $\alpha^3 = 5 - \sqrt{3}$, i.e., $\alpha^3 - 5 = -\sqrt{3}$, hence $(\alpha^3 - 5)^2 = 3$, i.e., $\alpha^6 - 10\alpha^3 + 22 = 0$. By the Rational Zeros Theorem 2.2 (page 9), the only *candidates* for a rational solution of the equation $x^6 - 10x^3 + 22 = 0$ are $x = \pm 1, \pm 2, \pm 11, \pm 22$. It is easy to check that none of these candidates is actually a solution of the equation. (For example, with $x = 2$ we have $x^6 - 10x^3 + 22 = 64 - 10 \cdot 8 + 22 = 6 \neq 0$.) Since α is a solution but is not among the candidates for a rational solution, it follows that α is not rational.

3.4 Part (v): By part (iv) we have $0 \leq a^2$ for all a. In particular, with $a = 1$ we have $0 \leq 1^2$. By M3, $1^2 = 1 \cdot 1 = 1$. Thus $0 \leq 1$. One of our axioms (stated in class) was that $0 \neq 1$. Hence $0 < 1$, Q.E.D.

Part (vii): Assume $0 < a < b$. By O3 it follows that $0 < b$ also. Applying part (vi) to a and b, it follows that $0 < a^{-1}$ and $0 < b^{-1}$. Multiplying the inequality $a < b$ by a^{-1} and using O5, we get $1 < a^{-1} \cdot b$. Multiplying by b^{-1} and using O5 again, we get $b^{-1} < a^{-1}$. This completes the proof.

Note: Some of our axioms and parts of theorems were stated with $\leq$, but we have applied them with $<$. This is justified, because in each case it is easy to prove that the version with $\leq$ implies the version with $<$.

3.7 (a) First assume $|b| < a$. By 3.2(i), $-a < -|b|$. By 3.5(i) $|b| \geq 0$, hence $-|b| \leq 0 \leq |b|$ by 3.2(i). Thus we have $-a < -|b| \leq |b| < a$. Since b is equal to either $|b|$ or $-|b|$, it follows that $-a < b < a$. Now conversely, assume $-a < b < a$. By 3.2(i) we have $-a < -b < a$ also. Since $|b|$ is equal to either b or $-b$, it follows that $-a < |b| < a$. This completes the proof.

(b) By part (a) we have $|a - b| < c$ if and only if $-c < a - b < c$. By O4 this is equivalent to $b - c < a < b + c$.

(c) We can redo parts (a) and (b) with $\leq$ replacing $<$ throughout. Then, the redone version of (b) says that $|a-b| \leq c$ if and only if $b-c \leq a \leq b+c$.