Math 312, Intro. to Real Analysis: Midterm Exam #1

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1. True or False (3 points each)

   (a) Every ordered field has the Archimedean property.

   (b) The ordered field axioms imply $|a - b| \leq |a| + |b|$ for all a, b.

   (c) If $\lim a_n = -\infty$ then $\limsup a_n = -\infty$.

   (d) For any sequence of real numbers, the lim inf and the lim sup always exist and furthermore the lim inf is always $\leq$ the lim sup.

   (e) The equation $3x^3 + 2x^2 + 3x + 2 = 0$ has a rational solution.

   (f) $\sqrt[3]{216}$ is an irrational number.

   (g) The limit of a convergent sequence of negative numbers is negative.

   (h) The limit of a convergent sequence of rational numbers is rational.

   (i) Every interval contains at least three rational numbers.

   (j) Every bounded sequence of real numbers is convergent.

   (k) Every convergent sequence of real numbers is bounded.

   (l) Every monotone sequence of real numbers is convergent.

   (m) If (a_n) is a monotone sequence of real numbers, then $\lim a_n$ exists and belongs to the interval $(-\infty, \infty)$.

2. (7 points each)

   (a) Give an example of a sequence of real numbers such that

   $$-\infty < \inf a_n < \lim a_n < \sup a_n < \infty.$$

   (b) Give an example of a sequence of real numbers such that

   $$\limsup a_n, \quad \liminf a_n, \quad \sup a_n, \quad \inf a_n$$

   are four distinct real numbers.

   (c) Give an example of a sequence of real numbers such that

   $$\liminf a_n = -\infty \qquad \text{and} \qquad \limsup a_n = \sqrt{2}.$$

3. (8 points) It can be shown that $\sqrt[3]{1+\sqrt{5}}$ is an *algebraic number*, i.e., it is a solution of some polynomial equation with integer coefficients. Find such an equation.

4. (8 points) Find all candidates for rational solutions of the equation

$$2x^2 - ax + 5 = 0$$

where a is an unspecified integer.

5. (12 points) Use algebra plus limit laws to calculate

$$\lim \frac{\sqrt{2n^2+5n}}{n+4}.$$

6. (12 points) It can be shown that

$$\lim \frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} = 1.$$

Given $\epsilon > 0$, find an N so large that

$$\left| \frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} - 1 \right| < \epsilon$$

holds for all $n > N$.