Math 312, Intro. to Real Analysis: Final Exam

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There are problems worth a total of 150 points.
Please be sure to attempt all problems.

1. True or false (3 points each).

   (a) For all sequences of real numbers (s_n) we have $\liminf s_n \leq \limsup s_n$.

   (b) Every bounded sequence of real numbers has at least one subsequential limit.

   (c) If the functions f_n are continuous on $[0,1]$ and converge uniformly to the function f on $[0,1]$, then f is uniformly continuous on $[0,1]$.

   (d) If the radius of convergence of a power series $\sum a_k x^k$ is R, and if $0 < R < \infty$, then the series $\sum a_k x^k$ converges uniformly on $(-R, R)$.

   (e) The integral of the limit is equal to the limit of the integrals.

2. (10 points) The real number system $\mathbb{R}$ has been characterized in terms of Axioms A1–A4, M1–M4, DL, O1–O5, and the Completeness Axiom. Which of these axioms fail for the rational number system $\mathbb{Q}$? Give one or more examples illustrating your answers.

3. (10 points)

   (a) State the formal definition of what it means for a sequence of real numbers (s_n) to converge to a limit s.

   (b) In terms of your definition from part (a), prove directly that

$$\lim_{n\to\infty} \frac{\pi}{\sqrt{n - 100\pi}} = 0.$$

4. (7 points) Calculate $\displaystyle\lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \cdots + (-1)^n \frac{3}{2^n} \right)$.

5. (a) (2 points) Give an example of a bounded sequence of real numbers with exactly two subsequential limits.

   (b) (2 points) Give an example of a bounded sequence of real numbers with exactly five subsequential limits.

   (c) (4 points) Give an example of a bounded sequence of real numbers with infinitely many subsequential limits.

6. (10 points) Give an example of a sequence of continuous functions on $[0, 1]$ such that $f_n \to 0$ pointwise but not uniformly on $[0, 1]$.

7. (15 points) Prove that if $\sum |a_k|$ is convergent then $\sum a_k$ is convergent.

   Hint: Use the Triangle Inequality.

8. (3 points each) For each of the following series, tell whether the series is convergent or divergent. State which convergence/divergence test you are using, and show any needed calculations.

   (a) $\sum \frac{1}{2^n}$

   (b) $\sum \frac{(-1)^n}{\sqrt{n}}$

   (c) $\sum (-1)^n \frac{n}{100n + 1000}$

   (d) $\sum \left(\frac{7n}{8n + 1} \right)^n$

   (e) $\sum \frac{1}{n \log n}$

9. (15 points) For each of the following functions, say whether the function is *continuous* and/or *uniformly continuous* on each of the three intervals

$$[0, 1], \qquad (0, 1), \qquad (2, \infty).$$

   You are not required to justify or prove your answers.

   (a) $\sin x$

   (b) e^x

   (c) $|x - \frac{1}{2}| + |x - 3|$

   (d) $\frac{1}{1 - x}$

   (e) $\sum_{n=0}^{\infty} x^n$

10. (10 points)

    (a) Prove directly that $x^n \to 0$ uniformly on the interval $[-0.99, 0.99]$.

    (b) Does $x^n \to 0$ uniformly on the interval $(-1, 1)$? Justify your answer.

11. (10 points)

    (a) Calculate $\lim_{n\to\infty} \int_{-1}^{1} \sin^n x \, dx$.

    (b) Justify your calculation for part (a) by stating an appropriate property of the functions $\sin^n x$ and an applicable theorem.

12. (15 points) For each of the following series, determine the set of all x such that the series converges at x. Show any needed calculations.

    (a) $\sum_{n=0}^{\infty} 2\cos^n x$

    (b) $\sum_{n=1}^{\infty} \frac{1}{nx^n}$

    (c) $\sum_{n=0}^{\infty} (2+\sqrt{n})x^n$

    (d) $\sum_{n=0}^{\infty} \frac{(10x)^n}{n^2+1}$

    (e) $\sum_{n=0}^{\infty} n(x-2)^n$

13. (10 points)

    (a) Determine the coefficients a_k for $k = 0, 1, 2, \ldots$ such that

    $$\sum_{k=0}^{\infty} a_k x^k = \frac{x^2+1}{x-1}. \tag{1}$$

    (b) Over what interval is the above equation (1) valid?