Final Exam – MATH 311W Section 003

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There are 7 problems. For each problem, please exhibit your work leading to the solution of the problem. Please write neatly and do not show scratch work.

1. Exhibit a finite state automaton M, F with the following property: if s is any finite sequence of letters from the alphabet $\{a, b, c\}$, then M, F accepts s if and only if each of the letters a, b, c occurs at least once in s.

2. Consider the permutations $\pi = (1\,2\,3)(4\,5\,6\,7\,8\,9)$ and $\sigma = (3\,4)$.

    (a) What is the sign of π?

    (b) What is the order of π?

    (c) What is the shape of π?

    (d) Exhibit the permutation π^{-1}.

    (e) Exhibit the permutation π^3.

    (f) Exhibit the permutations π^{43} and π^{48}.

    (g) Exhibit π as a product of transpositions.

    (h) Exhibit the permutation $\sigma^{-1}\pi\sigma$.

    (i) Exhibit the permutations $\pi\sigma$ and $\sigma\pi$.

    (j) Exhibit the cyclic decomposition of $\pi\sigma$.

3. True or false. Do not give reasons for your answers.

    (a) For all integers $n \geq 2$ and all integers a, the multiplicative order of a modulo n is a divisor of $\phi(n)$.

    (b) For all integers $n \geq 2$, the group $G_n = \mathbb{Z}_n^*$ is Abelian.

    (c) For all integers $n \geq 2$, the permutation group $S(n)$ is Abelian.

(d) If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ exists and is a bijection.

(e) The union of any two relations is a relation.

(f) Given a set X and an equivalence relation E on X, there is a canonical injection $\phi : X \to X/E$.

(g) For all finite sets X, Y, Z we have

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |X \cap Z| + |X \cap Y \cap Z|.$$

(h) Any permutation in $S(n)$ can be written as a product of disjoint transpositions.

(i) The inverse of any function is a relation.

(j) Every relation is the inverse of some function.

4. Let x be a real variable. Let f and g be the functions defined $f(x) = x^2$ and $g(x) = x + 5$.

   (a) What are the functions fg, gf, f^2, g^2, f^2g, g^2f, and f^2g^2?

   (b) What are the domains and ranges of fg and gf?

5. Define the concept of disjoint permutations. Prove that for any two disjoint permutations π and σ we have $\pi\sigma = \sigma\pi$.

6. (a) Define what is meant by the shape of a permutation.

   (b) Define what it means for two permutations to be conjugate.

   (c) Explain the relationship between these two concepts.

   (d) Illustrate your explanation with an example.

7. Prove that
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

holds for all positive integers n.