Solutions to graded exercises in Homework #8
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These exercises are from §2.9 in the textbook.

§2.9 Ex. 5. In general, the *coordinate vector* of a given vector $\mathbf{v}$ with respect to a given linearly independent set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is defined to be the unique vector $\mathbf{x}$ such that $[\,\mathbf{u}_1 \cdots \mathbf{u}_p\,]\,\mathbf{x} = \mathbf{v}$. For this exercise, row reduction shows that

$$\begin{bmatrix} 1 & -3 & 4 \\ 5 & -7 & 10 \\ -3 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.25 \\ 0 & 1 & -1.25 \\ 0 & 0 & 0 \end{bmatrix}$$

and this implies that the coordinate vector of $\begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix}$ with respect to the linearly independent set $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 0.25 \\ -1.25 \end{bmatrix}$.

§2.9 Ex. 11. Row reduction shows that

$$A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the pivot columns of A are columns 1, 2, 4. In other words, the column vectors

$$\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$$

form a basis of $\operatorname{Col} A$. This implies that $\dim \operatorname{Col} A = 3$ and $\dim \operatorname{Nul} A = 5 - 3 = 2$. To obtain a basis of $\operatorname{Nul} A$, perform further row reduction to get

$$A \sim U = \begin{bmatrix} 1 & 0 & -9 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as the reduced row echelon form of A. We know that $\operatorname{Nul} A$ is the solution set of $A\mathbf{x} = \mathbf{0}$ which is the same as the solution set of $U\mathbf{x} = \mathbf{0}$. The standard parametric description of this solution set is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_3 and x_5 are the free variables. Thus we see that the vectors

$$\begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

form a basis of $\operatorname{Nul} A$.

§2.9 Ex. 13. Let A be the matrix with the given vectors as its columns. By row reduction we have

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so columns 1, 3, 4 of A are the pivot columns of A. We conclude that these three columns of A are a basis of the subspace spanned by all the columns of A.

§2.9 Ex. 17. In general, for any $m \times n$ matrix A, the solution set of $A\mathbf{x} = \mathbf{0}$ is $\operatorname{Nul} A$, the null space of A, and its dimension is $n - \operatorname{rank}(A)$. In the case of a 7×6 matrix of rank 4, the dimension of $\operatorname{Nul} A$ is $n - \operatorname{rank}(A) = 6 - 4 = 2$.

§2.9 Ex. 21.

(a) True. This is just the definition of $[\mathbf{x}]_{\mathcal{B}}$, the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$.

(b) False. The 1-dimensional subspaces of $\mathbb{R}^n$ are the lines in $\mathbb{R}^n$ *which pass through the origin.*

(c) True. See Example 7 in §2.8 of the textbook.

(d) True. For any matrix A, the pivot columns of A form a basis for $\operatorname{Col} A$, and the non-pivot columns of A correspond to the free variables in the standard parametric description of $\operatorname{Nul} A$. See also Exercises 11 and 13 above, and Example 6 in §2.8, and Theorem 14 in §2.9.

(e) True. See Theorem 15 in §2.9.