Solutions to graded exercises in Homework #7
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Note: This assignment includes review exercises for the midterm exam.

§2.3 Ex. 7. Row reduction shows that the given 4×4 matrix has four pivot positions. Therefore, by Theorem 8, the matrix is invertible.

§2.8 Ex. 26. Row reduction shows that the pivot columns of A are

$$\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}$$

so these vectors form a basis of $\operatorname{Col} A$. Additional row reduction shows that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2.5 \\ 0 & 1 & 2 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the system $A\mathbf{x} = \mathbf{0}$ is equivalent to

$$\begin{array}{rrrrrcl} x_1 & & +3x_3 & & +2.5x_5 & = & 0 \\ & x_2 & +2x_3 & & +1.5x_5 & = & 0 \\ & & & x_4 & +x_5 & = & 0 \end{array}$$

with free variables x_3 and x_5. Therefore, the general solution of $A\mathbf{x} = \mathbf{b}$ may be parametrized as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

and the vectors $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ form a basis of $\operatorname{Nul} A$.

§1.3 Ex. 14. Row reduction shows that the pivot columns of the augmented matrix $[\,A \mid \mathbf{b}\,]$ are just the columns of A. Since $\mathbf{b}$ is not a pivot column, the system $A\mathbf{x} = \mathbf{b}$ is consistent. In other words, $\mathbf{b}$ is a linear combination of the columns of A, i.e., $\mathbf{b}$ belongs to $\operatorname{Col} A$.

Note: We can actually say more. Since all of the columns of A are pivot columns, $A\mathbf{x} = \mathbf{b}$ is consistent for *any* $\mathbf{b}$.

§1.7 Ex. 31. Since the third column of A is the sum of the first two columns, a nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

§1.8 Ex. 10. Row reduction shows that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the system $A\mathbf{x} = \mathbf{0}$ is equivalent to $x_1 + 3x_3 = x_2 + 2x_3 = x_4 = 0$ and the general solution may be parametrized as $\mathbf{x} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. Thus $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is a basis of $\operatorname{Nul} A$.