Solutions to graded exercises in Homework #14
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These exercises are from §§ 6.5 and 7.1 in the textbook.

§6.5 Ex. 4. We have

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

The normal system associated to $A\mathbf{x} = \mathbf{b}$ is $A^TA\mathbf{x} = A^T\mathbf{b}$, i.e.,

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

The augmented matrix of the normal system is

$$\begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

so the unique least-squares solution of $A\mathbf{x} = \mathbf{b}$ is $\widehat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

§6.5 Ex. 10. We have $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$. Since the columns of A are orthogonal, the projection of $\mathbf{b}$ to $\operatorname{Col} A$ is given by the formula

$$\widehat{\mathbf{b}} = \frac{\mathbf{a}_1 \cdot \mathbf{b}}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a}_2 \cdot \mathbf{b}}{\mathbf{a}_2 \cdot \mathbf{a}_2}\mathbf{a}_2 = \frac{9}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{12}{24} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$

The least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are the solutions of $A\mathbf{x} = \widehat{\mathbf{b}}$, i.e.,

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$

The augmented matrix of $A\mathbf{x} = \widehat{\mathbf{b}}$ is

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 4 & -1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the unique least-squares solution of $A\mathbf{x} = \mathbf{b}$ is

$$\widehat{\mathbf{x}} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}.$$

§7.1 Ex. 12. The given matrix

$$U = \begin{bmatrix} .5 & .5 & -.5 & -.5 \\ -.5 & .5 & -.5 & .5 \\ .5 & .5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$$

is easily seen to be *orthogonal*. (This means: the columns of U are orthogonal to each other, and the columns of U are unit vectors, and U is a square matrix.) Therefore, by Theorem 6 in §6.2 we have

$$U^{-1} = U^T = \begin{bmatrix} .5 & -.5 & .5 & -.5 \\ .5 & .5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}.$$

§7.1 Ex. 26.

(a) True. See Theorem 2 in §7.1.

(b) True, because then $B^T = (PDP^T)^T = P^{TT}D^TP^T = PDP^T = B$.

(c) False. For example, the 2×2 matrix $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal, but it is not symmetric, hence by Theorem 2 in §7.1 it is not orthogonally diagonalizable.

(d) True. See part b of Theorem 3 in §7.1.

§7.1 Ex. 34. The spectral decomposition of A is

$$A = 7\mathbf{u}_1\mathbf{u}_1^T + 7\mathbf{u}_2\mathbf{u}_2^T - 2\mathbf{u}_3\mathbf{u}_3^T$$

where

$$\mathbf{u}_1\mathbf{u}_1^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$\mathbf{u}_2\mathbf{u}_2^T = \frac{1}{18}\begin{bmatrix} 1 & -4 & -1 \\ -4 & 16 & 4 \\ -1 & 4 & 1 \end{bmatrix},$$

$$\mathbf{u}_3\mathbf{u}_3^T = \frac{1}{9}\begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}.$$