# Solutions to graded exercises in Homework #13

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These exercises are from §§ 6.2, 6.3, and 6.4 in the textbook.

§6.2 Ex. 20. The vectors

$$\mathbf{v}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$$

are orthogonal, because

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)(0) = 0.$$

On the other hand, while $\mathbf{v}_1$ is a unit vector, $\mathbf{v}_2$ is not. To normalize, replace $\mathbf{v}_2$ by the unit vector

$$\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{(\frac{1}{3})^2 + (\frac{2}{3})^2}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}.$$

§6.2 Ex. 24.
(a) False. An orthogonal set is always a linearly independent set.

(b) False. For an orthonormal set, the vectors are required to be unit vectors.

(c) True. See Theorem 7(a) in §6.2.

(d) True. The formula for the orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is

$$\frac{\mathbf{y} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

If we replace $\mathbf{v}$ by $c\mathbf{v}$, the $c$'s cancel out (provided $c \neq 0$).

(e) True. By definition, an *orthogonal matrix* is a square matrix with orthonormal columns. Since the columns are orthogonal, they are linearly independent, hence the matrix is invertible.

§6.3 Ex. 8. Let $W$ be the subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. We have

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = (1)(-1) + (1)(3) + (1)(-2) = 0$$

so $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$. Thus the orthogonal projection of $\mathbf{y}$ to $W$ is

$$\begin{aligned} \mathbf{y}_W &= \operatorname{proj}_W(\mathbf{y}) = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{7}{14}\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix}. \end{aligned}$$

Letting

$$\mathbf{z} = \mathbf{y} - \mathbf{y}_W = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} = \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix},$$

we have $\mathbf{y} = \mathbf{y}_W + \mathbf{z}$ where $\mathbf{y}_W$ is in $W$ and $\mathbf{z}$ is orthogonal to $W$.

§6.3 Ex. 16. Let $W$ be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, the orthogonal projection of $\mathbf{y}$ to $W$ is

$$\mathbf{y}_W = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{30}{10}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{26}{26}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}.$$

Letting

$$\mathbf{z} = \mathbf{y} - \mathbf{y}_W = \begin{bmatrix}3\\-1\\1\\13\end{bmatrix} - \begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix} = \begin{bmatrix}4\\4\\4\\4\end{bmatrix},$$

we see that the distance from $\mathbf{y}$ to $W$ is

$$\|\mathbf{y} - \mathbf{y}_W\| = \|\mathbf{z}\| = \sqrt{4^2 + 4^2 + 4^2 + 4^2} = 8.$$

§6.4 Ex. 10. The given column vectors are

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

The Gram-Schmidt process gives

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix},$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix},$$

and finally

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for the 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.