Solutions to graded exercises in Homework #12
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These exercises are from §§ 5.3, 6.1, and 6.2 in the textbook.

§5.3 Ex. 8. The matrix $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is triangular, so the eigenvalues are the diagonal entries $5, 5$. We then have $A - 5I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is of rank 1, which means that the dimension of the eigenspace is $2 - 1 = 1$. Thus A is not diagonalizable.

§5.3 Ex. 20. The matrix $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ is triangular, so the eigenvalues are the diagonal entries $4, 4, 2, 2$. For the eigenvalue 4 we have

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the general solution of $A\mathbf{x} = 4\mathbf{x}$ is

$$\mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and there are two linearly independent eigenvectors. For the eigenvalue 2 we have

$$A - 2I = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the general solution of $A\mathbf{x} = 2\mathbf{x}$ is

$$\mathbf{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and again there are two linearly independent eigenvectors. Thus A is diagonalizable and an explicit diagonalization is $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

§6.1 Ex. 6. We have

$$\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x} = \frac{(6)(3)+(-2)(-1)+(3)(-5)}{(6)(6)+(-2)(-2)+(3)(3)} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}.$$

§6.1 Ex. 20.
(a) True, because $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.
(b) False, because c could be negative.
(c) True, by definition of $W^\perp$.
(d) True, by Theorem 2 in §6.1.
(e) True, because the each entry of $A\mathbf{x}$ is the inner product of $\mathbf{x}$ with a row of A.

§6.2 Ex. 10. We have

$$\begin{aligned} \mathbf{u}_1\cdot\mathbf{u}_2 &= (3)(2)+(-3)(2)+(0)(-1) = 0 \\ \mathbf{u}_1\cdot\mathbf{u}_3 &= (3)(1)+(-3)(1)+(0)(4) = 0 \\ \mathbf{u}_2\cdot\mathbf{u}_3 &= (2)(1)+(2)(1)+(-1)(4) = 0 \end{aligned}$$

so $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form an orthogonal basis for $\mathbb{R}^3$. Therefore, $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ can be expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ by writing

$$\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{x}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 = \frac{24}{18}\mathbf{u}_1 + \frac{3}{9}\mathbf{u}_2 + \frac{6}{18}\mathbf{u}_3 = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3.$$