Solutions to graded exercises in Homework #11

Stephen G. Simpson

March 29, 2011

These exercises are from §§ 5.2 and 5.3 in the textbook.

§5.2 Ex. 8. The characteristic polynomial is

$$\begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda)-(-2)(2) = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2.$$

The eigenvalues are 5 and 5. In other words, 5 with multiplicity 2.

§5.2 Ex. 14. By cofactor expansion along the second row, the characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2-3\lambda-28) = (1-\lambda)(-4-\lambda)(7-\lambda) = -\lambda^3+4\lambda^2+25\lambda-28.$$

Although the eigenvalues are not required, we can easily find them by factoring the characteristic polynomial. The eigenvalues are $1, -4, 7$.

§5.2 Ex. 16. Since the matrix is triangular, the eigenvalues are the diagonal entries, namely $5, -4, 1, 1$.

§5.2 Ex. 20. We have

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det(A-\lambda I)^T = \det(A-\lambda I)$$

so the characteristic polynomial of $A^T$ is the same as that of $A$.

§5.3 Ex. 12. Let $A$ be the given matrix. Knowing that the eigenvalues of $A$ are 2 and 8, we can find the eigenvectors as follows. For $\lambda = 2$, row reduction gives

$$A - 2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the general solution of $A\mathbf{x} = 2\mathbf{x}$ is

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 8$, row reduction gives

$$A - 8I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the general solution of $A\mathbf{x} = 8\mathbf{x}$ is

$$\mathbf{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We have found three linearly independent eigenvectors, so the Diagonalization Theorem tells us that $A$ is diagonalizable. Using our eigenvectors as the columns of $P$ and the corresponding eigenvalues as the diagonal entries of $D$, we have

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

and an explicit diagonalization is $A = PDP^{-1}$.