Math 141H.1, Honors Calculus II

Bonus Problems 3

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Please work alone. You may use a calculator or computer algebra system if you wish, but please give exact solutions and show all the steps needed to obtain your solutions by hand.

Note: By a Maclaurin series, we mean a power series centered at $x = 0$.

1. Find the sum of the convergent series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

   Hint: Start with the Maclaurin series for $1/(1-x)$. Differentiate, multiply by x, differentiate, multiply by x, set $x = 1/2$.

2. Find the value of x for which $2^x + 2^{2x} + 2^{3x} + \cdots = 2$.

3. It can be shown that, for $|x| < 1$, the infinite product

   $$\prod_{n=1}^{\infty}(1+x^n) = (1+x)(1+x^2)(1+x^3)\cdots$$

   converges to a power series $1+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$. Find the first 12 coefficients: $a_1, a_2, \ldots, a_{12}$. Can you describe a_n combinatorially? Can you find a formula for a_n?

4. The *hyperbolic sine* and *hyperbolic cosine* functions are defined by

   $$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

   Find the Maclaurin series representations of $\sinh x$ and $\cosh x$. Show that the derivative of $\sinh x$ is $\cosh x$, and the derivative of $\cosh x$ is $\sinh x$. Show that $\sinh^2 x + 1 = \cosh^2 x$. Show that

   $$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$$

   and use this to find the Maclaurin series representation of $\sinh^{-1} x$. Show that

   $$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$$

5. Find the length of the parametrized curve $x = t\cos t$, $y = t\sin t$ for $0 \le t \le 2\pi$.