Math 141H.1, Honors Calculus II

Bonus Problems 2

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Please work alone. You may use a calculator or computer algebra system if you wish, but please give exact solutions and show all the steps needed to obtain your solutions by hand.

1. Find the absolute maximum value of the function $f(x) = (\sin x)^x$ over the interval $0 < x \leq \pi$. For which x is this achieved? Justify your answers.

2. The Fibonacci sequence is defined recursively by $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. The first few terms of the Fibonacci sequence are

   $$F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \ldots \;=\; 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

   and each term is the sum of the previous two terms.

   Assuming that

   $$\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

   exists and is finite, prove that this limit is exactly $(1+\sqrt{5})/2$. If you are unable to prove this, at least exhibit numerical values of F_n/F_{n-1} for $n = 1, \ldots, 20$ and compare them to $(1+\sqrt{5})/2$.

3. Find the length of the curve $y = \ln(1 - x^2)$ from $x = 0$ to $x = 1/2$.

   Note: The formula for the length of a curve $y = f(x)$ from $x = a$ to $x = b$ is

   $$L = \int_a^b \sqrt{1 + f'(x)^2}\; dx \;.$$

4. Determine which values of p have the following property: The area under the curve $y = x^{-p}$ from $x = 1$ to $x = \infty$ is infinite, but the volume of the solid generated by revolving this region about the x-axis is finite. (See Chapter 7, Additional Exercise 39, page 611.)

5. For each $x > 0$, the quantity $\Gamma(x)$ is defined by

   $$\Gamma(x) \;=\; \int_0^\infty t^{x-1}\, e^{-t}\, dt \;.$$

   Show that this improper integral is convergent. Show that $\Gamma(1) = 1$. Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. From these facts, deduce that $\Gamma(n) = (n-1)!$ for all $n = 1, 2, 3, \ldots$.