Peano systems

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1 Sets

A *set* is a collection of objects. The objects in the collection are called the *members* of the set, or the *elements* of the set. The notation $x \in A$ means that A is a set and x is one of the members of the set A.

To describe sets, we sometimes use "set-builder" notation such as $A = \{\ldots\}$. This means that A is a set and the members of A consist of all objects which have the property ....

Here are some examples.

1. $A = \{x, y, z\}$ means: A is the set consisting of x, y, and z.
2. $E = \{2, 4, 6, \ldots\}$ means: E is the set of even numbers.
3. $[1, 2) = \{x \mid 1 \leq x < 2\}$ means: $[1, 2)$ is the set consisting of all real numbers x such that $x \geq 1$ and $x < 2$.

Let A and B be sets. We say that A is a *subset* of B if every member of A is a member of B. Note that B itself is a subset of B. If A is a subset of B other than B itself, we say that A is a *proper subset* of B.

For example, E and $[1, 2)$ are proper subsets of $[1, \infty)$, while $[1, \infty)$ is an "improper" subset of $[1, \infty)$.

We write $A \subseteq B$ to mean that A is a subset of B. We write $A \subset B$ to mean that A is a proper subset of B. This kind of notation is analogous to the standard algebra notation $x \leq y$ (x is less than or equal to y) and $x < y$ (x is less than y).

The *extensionality principle* is a basic principle concerning sets. Namely, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. In other words, if A and B are sets and every member of A is a member of B and vice versa, then A and B are the same set.

For example $\{1, 2, 3\} = \{3, 2, 1\}$. Thus, a set does not depend on the order in which the elements of the set are listed.

2 Functions

A *function* is a rule which associates certain objects (the *values* of the function) to certain other objects (the *arguments* of the function). The expression $f(x) = y$ means that f is a function, x is one of the arguments of f, and y is the value associated to x by f. Think of the function f as a "black box" which accepts "inputs" and responds by producing "outputs." Thus, whenever you input the argument x, the box always outputs the value $f(x)$.

As an example, let f be the function defined by $f(x) = x^2$ for all real numbers x. We can then write $f(2) = 4$, etc.

Let f be any function. The *domain* of f is the set of arguments to which f assigns a value. The *range* of f is the set of values of f. Using set-builder notation, we may write

$$\mathrm{dom}(f) = \{x \mid f(x) \text{ is defined}\}$$

and

$$\mathrm{rng}(f) = \{y \mid y = f(x) \text{ for some } x \in \mathrm{dom}(f)\}.$$

Here are some examples.

1. The domain of the function $f(x) = x^2$ is $(-\infty, \infty)$ and the range is $[0, \infty)$.
2. The domain of the function $f(x) = \sqrt{1 - x^2}$ is $\mathrm{dom}(f) = [-1, 1]$ and the range is $\mathrm{rng}(f) = [0, 1]$.
3. Let f be the "brain" function, defined by $f(x) =$ the brain of x. Then presumably $\mathrm{dom}(f) = \{a \mid a \text{ is an animal}\}$ and $\mathrm{rng}(f) = \{b \mid b \text{ is the brain of some animal}\}$.

There is an *extensionality principle* for functions. Namely, if f and g are functions with the same domain, and if $f(x) = g(x)$ for all x belonging to the domain, then $f = g$. In other words, two functions which give the same values at the same arguments are actually the same function.

Let A and B be sets. We write $f : A \to B$ to mean that f is a function, $\mathrm{dom}(f) = A$, and $\mathrm{rng}(f) \subseteq B$. If $\mathrm{rng}(f) = B$ we say that f is *onto* B, or f *maps* A *onto* B. By definition, every function maps its domain onto its range.

A function is said to be *one-to-one* if it associates distinct values to distinct arguments. In other words, f is one-to-one if for all $x, y \in \mathrm{dom}(f)$, $x \neq y$ implies $f(x) \neq f(y)$. Examples:

1. The function $f(x) = \sqrt{x}$ is one-to-one.
2. The function $f(x) = x^2$ is not one-to-one.

3. The "brain" function is one-to-one.

Let A and B be sets. A *one-to-one correspondence* between A and B is a one-to-one function $f : A \to B$ which is onto B. In other words, each element of A corresponds to exactly one element of B and vice versa. Roughly speaking, the existence of a one-to-one correspondence means that A and B have the same "number" of elements. This concept applies even if the sets A and B are infinite, in which case it leads to a concept of "infinite numbers." This is the beginning of a mathematical subject known as *set theory*.

3 Systems

Our account of Peano systems is based on that of Mendelson [4, Chapter 2]. The original source of the material is Dedekind [2]; see also [3].

A *system* consists of a set A, an element $i \in A$, and a function $f : A \to A$. The element i is called the *initial element* of the system A, i, f.

If A, i, f and B, j, g are systems, we say that A, i, f is a *subsystem* of B, j, g provided $A \subseteq B$, $i = j$, and $f(x) = g(x)$ for all $x \in A$. Note that any system B, j, g is a subsystem of itself. A subsystem of B, j, g other than B, j, g itself is called a *proper subsystem* of B, j, g.

Exercises 1. Prove the following statements.

1. If A, i, f is a subsystem of B, j, g and B, j, g is a subsystem of C, k, h then A, i, f is a subsystem of C, k, h.
2. Given a system B, j, g we can find a subsystem of B, j, g which is a subsystem of every subsystem of B, j, g.
3. If A, i, f is a proper subsystem of B, j, g then A is a proper subset of B.
4. Given a system B, j, g we can find a subsystem of B, j, g which has no proper subsystem.

4 Peano systems

We define a *Peano system* to be a system A, i, f with the following properties.

1. f is one-to-one.
2. $i \notin \text{rng}(f)$.
3. The system A, i, f has no proper subsystem.

Exercise 2. Let B, j, g be a system with properties 1 and 2. Prove that B, j, g has a subsystem A, i, f which is a Peano system.

Let A, i, f be a system. A set $Z \subseteq A$ is said to be *inductive* (with respect to the system A, i, f) if (1) $i \in Z$, and (2) $f(x) \in Z$ whenever $x \in Z$. Note that the set A is itself inductive.

Exercise 3. Let A, i, f be a Peano system. Prove that the only inductive subset of A is A itself.

For any function f let us write $f^2(x) = f(f(x))$, and $f^3(x) = f(f(f(x)))$, and in general

$$f^n(x) = \underbrace{f(f(\ldots(f}_{n}(i)))$$

where n is any *natural number*, i.e., $n \in N$ where $N = \{1, 2, 3, \ldots\}$. It may be helpful to "visualize" property 3 by writing

$$A = \{i, f(i), f^2(i), f^3(i), \ldots, f^n(i), f^{n+1}(i), \ldots\}.$$

Note however that the natural number system, N, plays a key role in this method of visualization. For better or worse, our goal here is to explain number concepts in terms of the pure theory of sets and functions, with no prior reference to numbers. Thus, the above-mentioned visualization method is not strictly relevant in terms of our goal.

By this time the student would probably appreciate seeing an example of a Peano system. Assuming for a moment that we already understand the natural number system, a very good example of a Peano system is $N, 1, S$ where $S : N \to N$ is the *successor function*, $S(n) = n+1$ for all $n \in N$. Another example is $E, 2, g$ where E is the set of even numbers, $E = \{2, 4, 6, \ldots\}$, and $g : E \to E$ is defined by $g(n) = n + 2$ for all $n \in E$. Later we shall prove that all Peano systems are "essentially the same." More precisely, we shall prove that all Peano systems are *isomorphic* to each other.

5 Some theorems about Peano systems

Throughout this section A, i, f is assumed to be a Peano system. As already mentioned, our favorite example of a Peano system is $N, 1, S$ where $S : N \to N$ is given by $S(n) = n + 1$. However, our theorems will be stated and proved abstractly for any Peano system A, i, f.

Theorem 1. *Let A, i, f be a Peano system. Then, the range of f consists of all members of A other than i.*

In the case of $N, 1, S$ this means that $\text{rng}(S) = \{2, 3, 4, \ldots\}$.

Proof. Let Z be the subset of A consisting of i together all members of $\text{rng}(f)$. We claim that Z is an inductive set. To see this, note first that $i \in Z$, by definition of Z. In addition, for any $x \in Z$ we have $x \in A$ (since Z is a subset of A), hence $f(x) \in Z$ (since $f : A \to A$ and Z includes the range of f). We have now verified that Z is inductive. Since A, i, f is a Peano system, it follows that $A = Z$. In other words, every member of A other than i belongs to the range of f. Since the range of f is also included in A, it follows by extensionality that the range of f consists precisely of all members of A other than i. This completes the proof. □

Theorem 2. *Let A, i, f be a Peano system. Then, for all $x \in A$ we have $f(x) \neq x$.*

In the case of $N, 1, S$ this means that $S(n) \neq n$ for all $n \in N$.

Proof. Let Z be the subset of A consisting of all $x \in A$ such that $f(x) \neq x$. We claim that Z is an inductive set. To see this, note first that $f(i) \neq i$ (because $i \notin \text{rng}(f)$, by property 2 in the definition of a Peano system). In other words, $i \in Z$. In addition, for any $x \in B$ we have $f(x) \neq x$, hence $f(f(x)) \neq f(x)$ (because f is one-to-one, by property 1 in the definition of a Peano system), hence $f(x) \in Z$. Thus we have verified that Z is an inductive set. It follows that $A = Z$. In other words, any $x \in A$ belongs to Z, which means that $f(x) \neq x$. This completes the proof. □

Theorem 3 (Iteration Theorem)**.** *Let A, i, f be a Peano system, and let B, j, g be a system. Then, there is exactly one function $\Phi : A \to B$ defined by the following conditions: $\Phi(i) = j$, and $\Phi(f(x)) = g(\Phi(x))$ for all $x \in A$.*

As an example, consider the system $(-\infty, \infty), \sqrt{2}, g$ where $g(x) = x + \sqrt{2}$ for all real numbers x. Then, Theorem 3 tells us that there is a unique function $\Phi : A \to (-\infty, \infty)$ defined by $\Phi(i) = \sqrt{2}$ and $\Phi(f(x)) = \Phi(x) + \sqrt{2}$ for all $x \in A$. In the case of the Peano system $N, 1, S$ our function $\Phi : N \to (-\infty, \infty)$ can be described by writing $\Phi(n) = n\sqrt{2}$ for all $n \in N$.

As another example, consider the system $(-\infty, \infty), 1/2, h$ where $h(x) = x/2$ for all real numbers x. Then, Theorem 3 tells us that there is a unique function $\Phi : A \to (-\infty, \infty)$ such that $\Phi(i) = 1/2$ and $\Phi(f(x)) = \Phi(x)/2$ for all $x \in A$. In the case of the Peano system $N, 1, S$ our function $\Phi : N \to (-\infty, \infty)$ can be described by writing $\Phi(n) = 1/2^n$ for all $n \in N$.

We now proceed to prove Theorem 3.

Proof of Theorem 3. A function ϕ is said to be *admissible* if the following conditions hold.

1. $\text{dom}(\phi) \subseteq A$.
2. $i \in \text{dom}(\phi)$.
3. For all $u \in A$, if $f(u) \in \text{dom}(\phi)$ then $u \in \text{dom}(\phi)$.
4. $\text{rng}(\phi) \subseteq B$.
5. $f(i) = j$.
6. For all $u \in A$, if $f(u) \in \text{dom}(\phi)$ then $\phi(f(u)) = g(\phi(u))$.

Note that the first three conditions tell us something about the domain of ϕ, while the last three conditions tell us something about the values of ϕ. Theorem 3 amounts to saying that there is exactly one admissible function Φ such that $\text{dom}(\Phi) = A$.

Let x be any element of A. A function ϕ is said to be x-*admissible* if it is admissible and $x \in \mathrm{dom}(\phi)$.

As part of the proof of Theorem 3, we shall prove several claims.

Claim 1. If ϕ is $f(x)$-admissible then ϕ is x-admissible.

To see this, suppose ϕ is $f(x)$-admissible. Then clearly $f(x) \in \mathrm{dom}(\phi)$. Applying condition 3 with $u = x$, we see that $x \in \mathrm{dom}(\phi)$. In addition ϕ is admissible, so we have now shown that ϕ is x-admissible.

Claim 2. For each $x \in A$ there exists at least one x-admissible function.

To see this, let Z be the subset of A consisting of all $x \in A$ such that there exists at least one x-admissible function. We are going to show that Z is inductive. First, to show that $i \in B$, consider the function ϕ with domain $\{i\}$ such that $\phi(i) = j$. Clearly this function is i-admissible. Second, assume $x \in Z$. Then, by definition of Z, there exists an x-admissible function, call it ϕ. We know that $x \in \mathrm{dom}(\phi)$, but $f(x)$ may or may not belong to $\mathrm{dom}(\phi)$. If $f(x) \in \mathrm{dom}(\phi)$ then clearly ϕ is is $f(x)$-admissible. If $f(x) \notin \mathrm{dom}(\phi)$, we define another function ϕ^* as follows. Let $\phi^*(u) = \phi(u)$ for each $u \in \mathrm{dom}(\phi)$, and in addition let $\phi^*(f(x)) = g(\phi(x))$. Thus the domain of ϕ^* consists of the domain of ϕ plus the additional element $f(x)$. Clearly ϕ^* is $f(x)$-admissible, so in both cases there exists at least one $f(x)$-admissible function. We have now shown that Z is inductive. Hence, by property 3 in the definition of a Peano system, it follows that $Z = A$. In other words, for each $x \in A$ there exists at least one x-admissible function. This proves Claim 2.

Claim 3. For each $x \in A$, if ϕ_1 and ϕ_2 are x-admissible functions then $\phi_1(x) = \phi_2(x)$.

To see this, let Z be the subset of A consisting of all $x \in A$ such that $\phi_1(x) = \phi_2(x)$ for all x-admissible functions ϕ_1 and ϕ_2. We are going to show that Z is inductive. First, $i \in Z$ since $\phi_1(i) = j = \phi_2(i)$. Second, assume $x \in Z$ and let ϕ_1 and ϕ_2 be $f(x)$-admissible. It follows by Claim 1 that ϕ_1 and ϕ_2 are x-admissible. Since $x \in Z$ it follows that $\phi_1(x) = \phi_2(x)$. But then, by condition 6 in the definition of admissibility, it follows that $\phi_1(f(x)) = g(\phi_1(x)) = g(\phi_2(x)) = \phi_2(f(x))$. This shows that $f(x) \in Z$. We have now shown that Z is inductive. Hence, by property 3 in the definition of a Peano system, it follows that $Z = A$. In other words, for each $x \in A$ we have $\phi_1(x) = \phi_2(x)$ for all x-admissible functions ϕ_1 and ϕ_2. This proves Claim 3.

We can now define our function Φ as follows. Given $x \in A$, apply Claim 1 to choose an x-admissible function ϕ, and then let $\Phi(x) = \phi(x)$. By Claim 3 the value of $\Phi(x)$ does not depend on our choice of the x-admissible function ϕ. Thus we have defined a function $\Phi : A \to B$. We still need to show that Φ satisfies the conditions stated in Theorem 3.

Claim 4. $\Phi(i) = j$.

To see this, note that $\Phi(i) = \phi(i)$ for some i-admissible function ϕ. But then by condition 5 we have $\phi(i) = j$, hence $\Phi(i) = j$. This proves Claim 4.

Claim 5. For each $x \in A$ we have $\Phi(f(x)) = g(\Phi(x))$.

To see this, note that $\Phi(f(x)) = \phi(f(x))$ for some $f(x)$-admissible function ϕ. But then by condition 6 we have $\phi(f(x)) = g(\phi(x))$. Moreover, by Claim 1

ϕ is x-admissible, so $\Phi(x) = \phi(x)$. Putting all this together, we have $\Phi(f(x)) = \phi(f(x)) = g(\phi(x)) = g(\Phi(x)$. This proves Claim 5.

We have now shown that there exists a function $\Phi : A \to B$ as specified in Theorem 3. It remains to show that there is only one such function.

Assume that $\Phi_1 : A \to B$ and $\Phi_2 : A \to B$ are two such functions. We must show that $\Phi_1 = \Phi_2$. Let Z be the subset of A consisting of all $x \in A$ such that $\Phi_1(x) = \Phi_2(x)$. We are going to show that Z is inductive. First, $\Phi_1(i) = j = \Phi_2(i)$, hence $i \in Z$. Second, for any $x \in Z$ we have $\Phi_1(x) = \Phi_2(x)$, hence $\Phi_1(f(x)) = g(\Phi_1(x)) = g(\Phi_2(x)) = \Phi_2(f(x))$, hence $f(x) \in Z$. We have now shown that Z is inductive. Hence, by property 3 in the definition of a Peano system, it follows that $Z = A$. In other words, for each $x \in A$ we have $\Phi_1(x) = \Phi_2(x)$. Since $A = \mathrm{dom}(\Phi_1) = \mathrm{dom}(\Phi_2)$ it follows that $\Phi_1 = \Phi_2$.

This completes the proof of Theorem 3. □

6 Isomorphism of Peano systems

Two systems A, i, f and B, j, g are said to be *isomorphic* if there exists a one-to-one correspondence $\Phi : A \to B$ such that

1. $\Phi(i) = j$, and
2. $\Phi(f(x)) = g(\Phi(x))$ for all $x \in A$.

The idea here is that isomorphic systems are, in abstract mathematical terms, "identical."

Theorem 4. *Let A, i, f and B, j, g be Peano systems. Then A, i, f and B, j, g are isomorphic.*

Proof. By Theorem 3 we have a unique function $\Phi : A \to B$ defined by the equations $\Phi(i) = j$ and $\Phi(f(x)) = g(\Phi(x))$. It remains to show that Φ is a one-to-one correspondence between A and B.

Claim 1. $\Phi : A \to B$ is onto B.

To see this, consider $\mathrm{rng}(\Phi)$ as a subset of B. Note first that $j \in \mathrm{rng}(\Phi)$ since $j = \Phi(i)$. Furthermore, assuming $y \in \mathrm{rng}(\Phi)$ we have $y = \Phi(x)$ for some $x \in A$, hence $g(y) = g(\Phi(x)) = \Phi(f(x))$ so $g(y) \in \mathrm{rng}(\Phi)$. We have now shown that $\mathrm{rng}(\Phi)$ is inductive with respect to the Peano system B, j, g. It follows by property 3 of Peano systems that $\mathrm{rng}(\Phi) = B$. In other words, Φ is onto B. This completes the proof of Claim 1.

Claim 2. $\Phi : A \to B$ is one-to-one.

To see this, let Z be the subset of A consisting of all $x \in A$ such that $\Phi(v) \neq \Phi(x)$ for all $v \neq x$ belonging to A. We shall show that Z is inductive with respect to the Peano system A, i, f. First, to show that $i \in Z$, consider any $v \neq i$ in A. Theorem 1 tells us that $v = f(u)$ for some u in A, and then $\Phi(v) = \Phi(f(u)) = g(\Phi(u)) \neq j = \Phi(i)$ since $j \notin \mathrm{rng}(g)$. Since this holds for all $v \neq i$ in A, we see that $i \in Z$. Second, suppose $x \in Z$. We need to show that $f(x) \in Z$, i.e., $\Phi(v) \neq \Phi(f(x))$ for all $v \neq f(x)$ in A. There are two possibilities for v, namely $v = i$ and $v \neq i$. If $v = i$ we have $\Phi(v) = \Phi(i) = j \neq g(\Phi(x)) = \Phi(f(x))$ since $j \notin \mathrm{rng}(g)$. If $v \neq i$ Theorem 1 tells us that $v = f(u)$ for some $u \in A$, and then $f(u) = v \neq f(x)$, hence $u \neq x$, hence $\Phi(u) \neq \Phi(x)$ since $x \in Z$, hence $\Phi(v) = \Phi(f(u)) = g(\Phi(u)) \neq g(\Phi(x)) = \Phi(f(x))$ since g is one-to-one, so $\Phi(v) \neq \Phi(f(x))$. Thus in both cases we see that $\Phi(v) \neq \Phi(f(x))$. Since this holds for all $v \neq f(x)$ in A, we see that $f(x) \in Z$. We have now shown that Z is inductive. It follows by property 3 of Peano systems that $Z = A$. In other words, $\Phi(x) \neq \Phi(v)$ for all $x \in A$ and $v \in A$ such that $x \neq v$. Thus Φ is one-to-one. This completes the proof of Claim 2.

We have now shown that $\Phi : A \to B$ is onto B and one-to-one. In other words, Φ is a one-to-one correspondence between A and B. It follows that the Peano systems A, i, f and B, j, g are isomorphic, Q.E.D. $\square$

By Theorem 4 we see that all Peano systems are isomorphic to each other and to the standard Peano system $N, 1, S$. In this way the defnition of Peano systems emerges as a precise, abstract characterization of the essential, structural features of the natural number system.

7 Addition in Peano systems

Let A, i, f be a Peano system. For each $x \in A$ we apply Theorem 3 to the system $A, f(x), f$. In this way we obtain a function $\Phi_x : A \to A$ with the properties

1. $\Phi_x(i) = f(x)$, and
2. $\Phi_x(f(u)) = f(\Phi_x(u))$ for all $u \in A$.

We then introduce a binary operation $+$ defined by $x + u = \Phi_x(u)$. Using this notation, the above properties may be rewritten as

1. $x + i = f(x)$, and
2. $x + f(u) = f(x + u)$.

The purpose of this section is to prove that the binary operation $+$ enjoys the familiar attributes of addition. For example, we are going to prove a theorem stating that $x + u = u + x$ for all $x, u \in A$.

Of course one would expect addition to make sense in any Peano system, keeping in mind that our standard example of a Peano system is $N, 1, S$ where $S(n) = n + 1$. For example, properties 1 and 2 above imply that $f(i) + f(i) = f(f(i) + i) = f(f(f(i)))$. In the case of the Peano system $N, 1, S$ this means that $S(1) + S(1) = S(S(S(1)))$, or in other words, $2 + 2 = 4$. In a similar way, properties 1 and 2 imply

$$f(f(i)) + f(f(f(f(i)))) = f(f(f(f(f(f(f(i)))))))$$

and this is the Peano system version of $3 + 5 = 8$, etc.

Throughout this section it is assumed that A, i, f is a Peano system and $x, u, v, \ldots$ are elements of A.

Theorem 5. $x + (u + v) = (x + u) + v$.

Proof. FIx x and u. Consider the set $Z = \{v \mid x+(u+v) = (x+u)+v\}$. We are going to show that Z is inductive. First, we have $x+(u+i) = x+f(u) = f(x+u) = (x+u)+i$ so $i \in Z$. Second, for each $v \in Z$ we have $x+(u+v) = (x+u)+v$, hence $x+(u+f(v)) = x+f(u+v) = f(x+(u+v)) = f((x+u)+v) = (x+u)+f(v)$ so $f(v) \in Z$. We have now shown that Z is inductive. Since A, i, f is a Peano system, it follows that $Z = A$. In other words, $x + (u + v) = (x + u) + v$ for all $v \in A$. This holds for all $x, u \in A$, so the proof of Theorem 5 is now complete. □

In order to prove the next theorem, we first prove the following lemma.

Lemma 1.

1. $x + i = i + x$.

2. $f(x) + u = f(x + u)$.

Proof. To prove part 1, let $Z = \{x \mid x + i = i + x\}$. We shall show that Z is inductive. First, $i + i = i + i$ so $i \in Z$. Second, for each $x \in Z$ we have $x + i = i + x$, and hence by Theorem 5 we have $f(x) + i = (x + i) + i = (i + x) + i = i + (x + i) = i + f(x)$, so $f(x) \in Z$. We have now shown that Z is inductive. It follows that $Z = A$, in other words $x + i = i + x$ for all $x \in A$. Thus we have proved part 1. Now apply Theorem 5 plus part 1 to get $f(x) + u = (x + i) + u = x + (i + u) = x + (u + i) = (x + u) + i = f(x + u)$. This proves part 2. □

Theorem 6. $x + u = u + x$.

Proof. Fix x, and let $Z = \{u \mid x + u = u + x\}$. We are going to show that Z is inductive. First, part 1 of Lemma 1 tells us that $i \in Z$. Second, given $u \in Z$ we have $x + u = u + x$, hence by part 2 of Lemma 1 we have $x + f(u) = f(x + u) = f(u + x) = f(u) + x$, so $f(u) \in Z$. We have now shown that Z is inductive. It follows that $Z = A$, i.e., $x + u = u + x$ for all $u \in A$. This holds for all $x \in A$, so we have proved Theorem 6. □

Theorem 7. $u + x = v + x$ *implies* $u = v$.

Proof. Fix u, v such that $u \neq v$. Let $Z = \{x \mid u + x \neq v + x\}$. Using the fact that f is one-to-one, we shall show that Z is inductive. First we have $u+i = f(u) \neq f(v) = v+i$ so $i \in Z$. Second, given $x \in Z$ we have $u+x \neq v+x$, hence $u + f(x) = f(u + x) \neq f(v + x) = v + f(x)$ so $f(x) \in Z$. We have now shown that Z is inductive. It follows that $Z = A$, i.e., $u + x \neq v + x$ for all x. We have now proved that $u \neq v$ implies $u + x \neq v + x$. This is equivalent to Theorem 7, so we have proved Theorem 7. □

Theorem 8. $x + u \neq u$.

Proof. Fix x, and let $Z = \{u \mid x + u \neq u\}$. We shall show that Z is inductive. First, since $i \notin \text{rng}(f)$ we have $x + i = f(x) \neq i$, so $i \in Z$. Second, given $u \in Z$ we have $x + u \neq u$, hence $x + f(u) = f(x + u) \neq f(u)$ since f is one-to-one. Thus $f(u) \in Z$. We have now shown that Z is inductive. It follows that $Z = A$, i.e., $x + u \neq u$ for all u. This completes the proof. □

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[2] Richard Dedekind. *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig, 1888. XIX + 58 pages.

[3] Richard Dedekind. *Essays on the Theory of Numbers.* Dover, New York, 1963. V + 115 pages, translation of [1, 2].

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