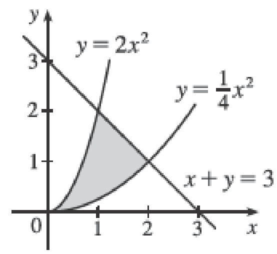
28. $\frac{1}{4}x^2 = -x+3 \quad\Leftrightarrow\quad x^2+4x-12=0 \quad\Leftrightarrow\quad (x+6)(x-2)=0 \quad\Leftrightarrow\quad x=-6$ or 2 and $2x^2=-x+3 \quad\Leftrightarrow$

$2x^2+x-3=0 \quad\Leftrightarrow\quad (2x+3)(x-1)=0 \quad\Leftrightarrow\quad x=-\frac{3}{2}$ or 1, so for $x\geq 0$,

$$A=\int_0^1\left(2x^2-\tfrac{1}{4}x^2\right)dx+\int_1^2\left[(-x+3)-\tfrac{1}{4}x^2\right]dx$$
$$=\int_0^1 \tfrac{7}{4}x^2\,dx+\int_1^2\left(-\tfrac{1}{4}x^2-x+3\right)dx$$
$$=\left[\tfrac{7}{12}x^3\right]_0^1+\left[-\tfrac{1}{12}x^3-\tfrac{1}{2}x^2+3x\right]_1^2$$
$$=\tfrac{7}{12}+\left(-\tfrac{2}{3}-2+6\right)-\left(-\tfrac{1}{12}-\tfrac{1}{2}+3\right)=\tfrac{3}{2}$$



52.

(a) We want to choose a so that

$$\int_1^a \frac{1}{x^2}\,dx=\int_a^4\frac{1}{x^2}\,dx \quad\Rightarrow\quad \left[\frac{-1}{x}\right]_1^a=\left[\frac{-1}{x}\right]_a^4 \quad\Rightarrow\quad -\frac{1}{a}+1=-\frac{1}{4}+\frac{1}{a} \quad\Rightarrow\quad \frac{5}{4}=\frac{2}{a} \quad\Rightarrow\quad a=\frac{8}{5}.$$

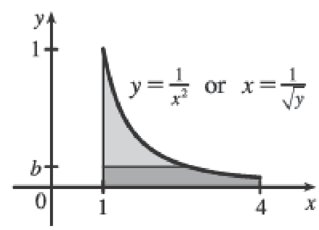
(b) The area under the curve $y=1/x^2$ from $x=1$ to $x=4$ is $\frac{3}{4}$ [take $a=4$ in the first integral in part (a)]. Now the line $y=b$ must intersect the curve $x=1/\sqrt{y}$ and not the line $x=4$, since the area under the line $y=1/4^2$ from $x=1$ to $x=4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y=1/x^2$ from $x=1$ to $x=4$. This implies that

$\int_b^1\left(1/\sqrt{y}-1\right)dy=\frac{1}{2}\cdot\frac{3}{4} \quad\Rightarrow\quad \left[2\sqrt{y}-y\right]_b^1=\frac{3}{8} \quad\Rightarrow\quad 1-2\sqrt{b}+b=\frac{3}{8} \quad\Rightarrow$

$b-2\sqrt{b}+\frac{5}{8}=0$. Letting $c=\sqrt{b}$, we get $c^2-2c+\frac{5}{8}=0 \quad\Rightarrow$

$8c^2-16c+5=0$. Thus, $c=\frac{16\pm\sqrt{256-160}}{16}=1\pm\frac{\sqrt{6}}{4}$. But $c=\sqrt{b}<1 \quad\Rightarrow$

$c=1-\frac{\sqrt{6}}{4} \quad\Rightarrow\quad b=c^2=1+\frac{3}{8}-\frac{\sqrt{6}}{2}=\frac{1}{8}\left(11-4\sqrt{6}\right)\approx 0.1503.$

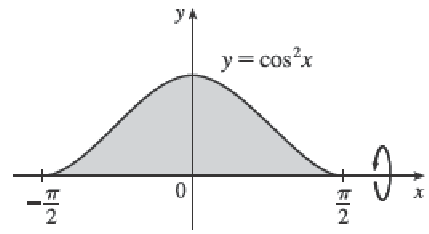


30. $\mathcal{R}_3$ about BC (the line $y=1$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\pi\left[(1-x)^2-\left(1-\sqrt[4]{x}\right)^2\right]dx=\pi\int_0^1\left[(1-2x+x^2)-(1-2x^{1/4}+x^{1/2})\right]dx$$
$$=\pi\int_0^1\left(x^2-2x-x^{1/2}+2x^{1/4}\right)dx=\pi\left[\tfrac{1}{3}x^3-x^2-\tfrac{2}{3}x^{3/2}+\tfrac{8}{5}x^{5/4}\right]_0^1=\pi\left(\tfrac{1}{3}-1-\tfrac{2}{3}+\tfrac{8}{5}\right)=\tfrac{4}{15}\pi$$

Note: See the note in Exercise 27. For Exercises 22, 26, and 30, we have $\frac{2}{3}\pi+\frac{1}{15}\pi+\frac{4}{15}\pi=\pi$.

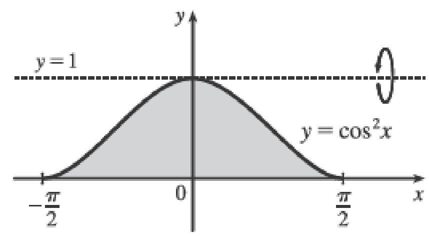
32. (a) About the x-axis:



$$V = \int_{-\pi/2}^{\pi/2} \pi(\cos^2 x)^2\,dx = 2\pi \int_0^{\pi/2} \cos^4 x\,dx \quad \text{[by symmetry]}$$

$$\approx 3.70110$$

(b) About $y = 1$:

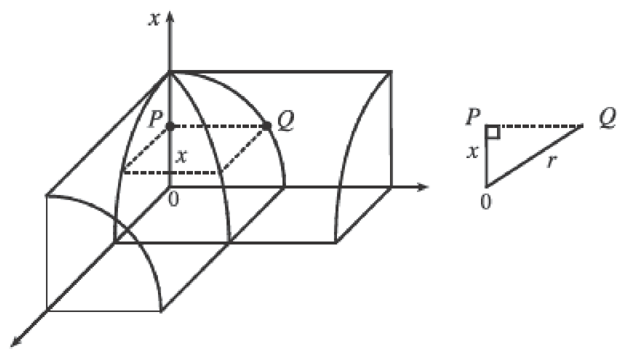


$$V = \int_{-\pi/2}^{\pi/2} \pi[(1-0)^2 - (1-\cos^2 x)^2]\,dx$$

$$= 2\pi \int_0^{\pi/2} [1 - (1 - 2\cos^2 x + \cos^4 x)]\,dx$$

$$= 2\pi \int_0^{\pi/2} (2\cos^2 x - \cos^4 x)\,dx \approx 6.16850$$

64. Each cross-section of the solid S in a plane perpendicular to the x-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2 = r^2 - x^2$. Therefore, $A(x) = 4(r^2 - x^2)$ and the volume of S is



$$V = \int_{-r}^{r} A(x)\,dx = 4\int_{-r}^{r}(r^2 - x^2)\,dx$$

$$= 8(r^2 - x^2)\,dx = 8\left[r^2x - \tfrac{1}{3}x^3\right]_0^r = \tfrac{16}{3}r^3$$

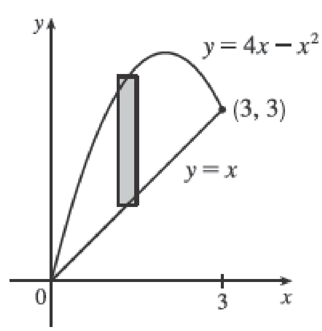
6. $4x - x^2 = x \quad \Leftrightarrow \quad 0 = x^2 - 3x \quad \Leftrightarrow \quad 0 = x(x-3) \quad \Leftrightarrow \quad x = 0$ or 3.

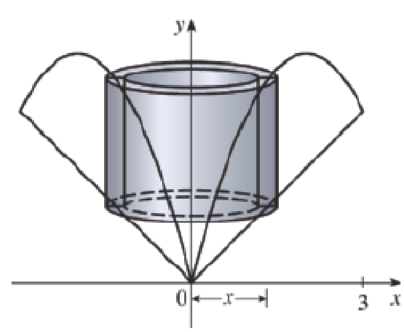
$$V = \int_0^3 2\pi x[(4x - x^2) - x]\,dx$$

$$= 2\pi \int_0^3 (-x^3 + 3x^2)\,dx$$

$$= 2\pi\left[-\tfrac{1}{4}x^4 + x^3\right]_0^3$$

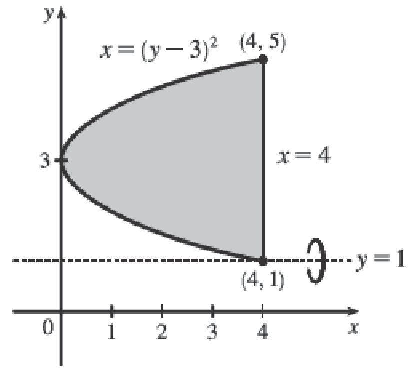
$$= 2\pi\left(-\tfrac{81}{4} + 27\right) = 2\pi\left(\tfrac{27}{4}\right) = \tfrac{27}{2}\pi$$





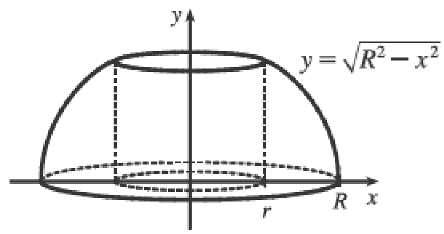
42. Use shells:

$$V = \int_1^5 2\pi(y-1)[4-(y-3)^2]\,dy$$
$$= 2\pi \int_1^5 (y-1)(-y^2+6y-5)\,dy$$
$$= 2\pi \int_1^5 (-y^3+7y^2-11y+5)\,dy$$
$$= 2\pi \left[-\tfrac{1}{4}y^4 + \tfrac{7}{3}y^3 - \tfrac{11}{2}y^2 + 5y\right]_1^5$$
$$= 2\pi\left(\tfrac{275}{12} - \tfrac{19}{12}\right) = \tfrac{128}{3}\pi$$



48.

By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x-axis and below the curve $y = \sqrt{R^2 - x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y-axis. This volume is equal to



$$2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi r h\,dx = 2\cdot 2\pi \int_r^R x\sqrt{R^2-x^2}\,dx = 4\pi\left[-\tfrac{1}{3}\left(R^2-x^2\right)^{3/2}\right]_r^R = \tfrac{4}{3}\pi(R^2-r^2)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = \left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 5.2.68.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 5.2.49,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \tfrac{4}{3}\pi R^3 - \pi r^2 h - 2\cdot\tfrac{\pi}{3}\left(R-\tfrac{1}{2}h\right)^2\left[3R - \left(R - \tfrac{1}{2}h\right)\right] = \tfrac{1}{6}\pi h^3$$